Hamilton equations for a particle subject to a central force

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A simple example that shows how to apply the Hamiltonian formalism in a case in which there is more than one generalized coordinate, is a particle subject to a central force. As it is often the case with simple examples, this problem can be easily treated both with the Hamiltonian and with the Lagrangian formalism. In this example the formalisms are equivalent and one cannot find one that is clearly better than the other.

Because of the conservation of angular momentum, the motion must take place on a plane. It is convenient to describe the motion on that plane in terms of polar coordinates, centered in the center of the force (remember that we are assuming that the force is central). The Hamiltonian is then

$$H = T + U(r) \qquad \text{with} \quad T = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2)$$

The Hamiltonian must be written in terms of the generalized momenta

$$p_r = \frac{\partial T}{\partial \dot{r}} = \underbrace{m\dot{r}}_{\text{LINEAR MOMENTUM}} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \underbrace{m r^2 \dot{\phi}}_{\text{ANGULAR MOMENTUM}}$$

$$\boxed{\dot{r} = \frac{p_r}{m}} \qquad \boxed{\dot{\phi} = \frac{p_\phi}{m r^2}}$$

therefore

$$\boxed{H = T + U(r) = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r)}$$

Hamilton equations are

$$\frac{\partial H}{\partial p_r} = \underbrace{\frac{p_r}{m} = \dot{r}}_{\text{definition of linear momentum}} \qquad \frac{\partial H}{\partial r} = -\frac{p_\phi^2}{mr^3} + \frac{dU}{dr} = -\dot{p}_r$$

$$\dot{p}_r = m\ddot{r} = \frac{p_\phi^2}{mr^3} - \frac{dU}{dr}$$

'radial' Newton's 2nd law

$$\frac{\partial H}{\partial p_\phi} = \underbrace{\frac{p_\phi}{mr^2} = \dot{\phi}}_{\text{definition of angular momentum}} \qquad \frac{\partial H}{\partial \phi} = \underbrace{0 = -\dot{p}_\phi}_{\text{conservation of angular momentum}}$$