Lagrange equations: One particle in 2D, polar coordinates

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As a first application of Lagrange equations, let's try to describe a particle free to move in two dimensions and subject to a conservative force determined by a potential U. In addition, let's describe the position and velocity of the particle in polar coordiantes. The Lagrangian will therefore be

$$\mathcal{L}(r,\phi,\dot r\,\dot\phi) = \underbrace{\frac{1}{2} m(\dot r^2 + r^2\dot\phi^2)}_{T} - U(r,\phi)$$

One can write two Lagrange's equations, one by taking derivatives with respect to r, the other by taking derivatives with respect to Φ

$$\boxed{\frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot r} = 0} \quad \text{eq (I)}$$

$$\boxed{\frac{\partial\mathcal{L}}{\partial\phi} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot\phi} = 0} \quad \text{eq (II)}$$

First equation:

$$\frac{\partial\mathcal{L}}{\partial r} = m r\dot\phi^2 - \frac{\partial U}{\partial r} \qquad \frac{\partial\mathcal{L}}{\partial\dot r} = m\dot r$$

$$\frac{\partial\mathcal{L}}{\partial r} - \frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot r} = m r\dot\phi^2 - \frac{\partial U}{\partial r} - m\ddot r = 0$$

$$m\ddot r = -\frac{\partial U}{\partial r} + m r\dot\phi^2$$

radial acceleration

$$m\ddot r - m\underbrace{r\dot\phi^2}_{} = \underbrace{-\frac{\partial U}{\partial r}}_{}$$

$m\frac{v_T^2}{r}$ = centripetal acceleration F_r = radial force

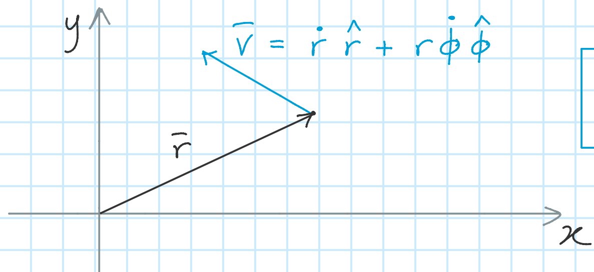
This equation reproduces Newton's second law along the radial direction.

Second equation

$$\frac{\partial \mathcal{L}}{\partial \phi} = -\frac{\partial U}{\partial \phi} \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = m r^2 \dot{\phi}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -\frac{\partial U}{\partial \phi} - \frac{d}{dt}\left(m r^2 \dot{\phi}\right) = 0$$

One can then recognize that the time derivative is indeed applied to the angular momentum of the particle calculated with respect to the origin of the frame of reference



$$\boxed{L = m r^2 \dot{\phi}}$$

It is useful to keep in mind the equation for the gradient in polar coordinates:

$$\nabla U = \underbrace{\frac{\partial U}{\partial r}}_{-F_r} \hat{r} + \underbrace{\frac{1}{r}\frac{\partial U}{\partial \phi}}_{-F_\phi} \hat{\phi}$$

Therefore the second equation can be rewritten as

$$\frac{d}{dt} L = r F_\phi \equiv \tau \longleftarrow \text{TORQUE}$$

Also the second equation corresponds to something that is known in Newtonian mechanics, namely that the torque applied to a particle corresponds to the rate of change of angular momentum. In addition, we can readily see that if the potential U does not depend on Φ, then the angular momentum L is conserved. This is indeed a general result, if the Lagrangian does not depend on a given generalized coordinate, then the associated generalized momentum is conserved

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} - \underbrace{\frac{\partial \mathcal{L}}{\partial q_i}}_{\text{assume is zero}} = 0 \longrightarrow \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \text{constant}$$