Linear air resistance

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Here we want to study the case in which the term linear in the velocity dominates the air drag. Mathematically, this is the case that leads to the equations of motion that can be solved most easily

linear air drag

$$m\bar{a} = \bar{F} \longrightarrow m\ddot{\bar{r}} = m\bar{g} - b\bar{v}$$

Since there is no explicit dependence on r, the equation above can be interpreted as a first order differential equation for the velocity v.

$$m\dot{\bar{v}} = m\bar{g} - b\bar{v} \xrightarrow[\text{in components}]{} \begin{cases} m\dot{v}_x = -b v_x \\ m\dot{v}_y = -mg - b v_y \end{cases}$$

The equations for the x and y components of the velocity are uncoupled (decoupled): One equation involves only the x component of the velocity, the other equation involves only the y component of the velocity. This is the feature that makes these equations easy to solve.

Notice that if the quadratic term in the drag force dominates, the equations do not decouple:

$$m\dot{\bar{v}} = m\bar{g} - c v^2 \hat{v} = m\bar{g} - c v \bar{v} \longrightarrow \begin{cases} m\dot{v}_x = -c\sqrt{v_x^2 + v_y^2}\, v_x \\ m\dot{v}_y = -mg - c\sqrt{v_x^2 + v_y^2}\, v_y \end{cases}$$

Equation for the x component

We want to solve the equation for the horizontal component of the velocity

$$\boxed{m\dot{v}_x = -b v_x}$$

This equation alone describes for example the motion of a cart moving on a horizontal frictionless track in a fluid that applies a linear drag force.

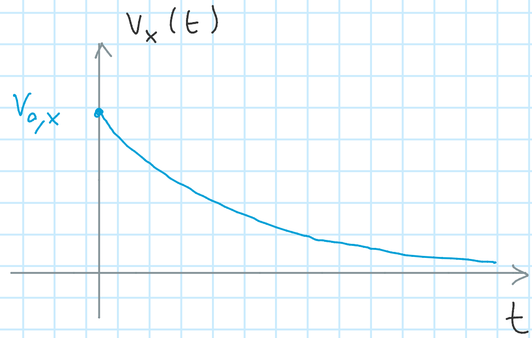
$$\frac{dv_x}{dt} = -\frac{b}{m} v_x \longrightarrow \boxed{v_x(t) = A\, e^{-\frac{b}{m}t}}$$

In the general solution above, A is a constant that must be fixed by specifying what is the velocity of the object at the time t=0.

$$v_x(t) = v_{0,x}\, e^{-\frac{b}{m}t}$$

Notice that

$$\lim_{t\to\infty} v_x(t) = 0$$



One can also easily find the dependence of x on t

$$v_x = \frac{dx}{dt} \qquad x(t) = \int_0^t v_x(t')\,dt' + x_0$$

initial x coordinate of the object

$$x(t) = v_{0,x}\int_0^t e^{-\frac{b}{m}t'}\,dt' + x_0$$

$$= v_{0,x}\left(-\frac{m}{b}\right)\left[e^{-\frac{b}{m}t'}\right]_0^t + x_0$$

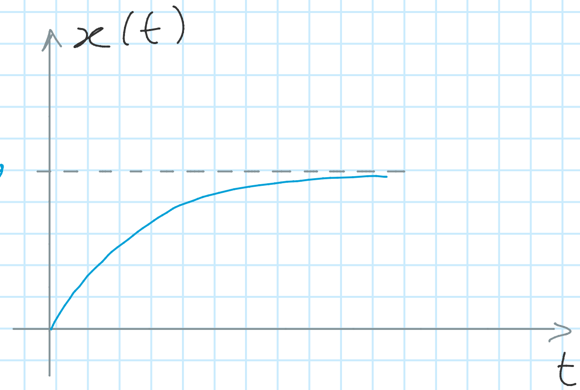
$$= -v_{0,x}\frac{m}{b}\left(e^{-\frac{b}{m}t} - 1\right) + x_0$$

$$\boxed{x(t) = v_{0,x}\frac{m}{b}\left(1 - e^{-\frac{b}{m}t}\right) + x_0}$$

If the object starts its motion from $x = 0$, the equation for x simplifies to

$$x(t) = v_{0,x}\,\tau\left(1 - e^{-\frac{t}{\tau}}\right) \qquad \tau \equiv \frac{m}{b} \qquad [\tau] = s$$

$$\lim_{t\to\infty} x(t) = v_{0,x}\,\tau \equiv x_\infty$$

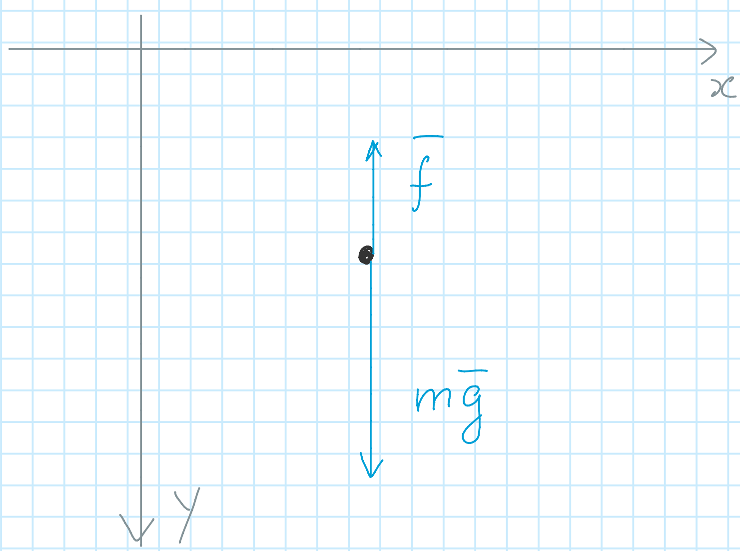


Equation for the y component

Now we want to solve the equation for the vertical component of the velocity

$$\boxed{m\dot{v}_y = mg - b v_y}$$

Notice that here we reversed the direction of the y axis with respect to the beginning of this set of notes. The y axis in the equation above points downwards, so that there is a plus sign in front of mg. This is more convenient to describe objects that are thrown vertically downward.



The projectile's velocity will increase until the velocity is such that the linear air drag cancels the weight

$$mg = b\, v_{ter} \qquad \boxed{v_{ter} = \frac{mg}{b}}$$

TERMINAL VELOCITY

Consequently, the equation for the y component of the velocity can be written as

$$m\dot{v}_y = b\left(\frac{mg}{b} - v_y\right) = b\left(v_{ter} - v_y\right)$$

To solve this equation let's introduce the new variable

$$u \equiv v_y - v_{ter} \qquad \dot{u} = \dot{v}_y$$

$$m\dot{u} = -b\,u \longrightarrow u = A e^{-\frac{b}{m}t}$$

$$v_y = v_{ter} + A e^{-\frac{b}{m}t}$$

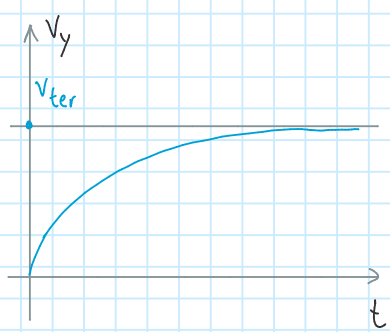
The constant A can be fixed by imposing that the equation returns the initial velocity along y at t = 0

$$v_y(t=0) \equiv v_{0,y} = v_{ter} + A \qquad A = v_{0,y} - v_{ter}$$

$$\boxed{v_y(t) = v_{ter} + \left(v_{0,y} - v_{ter}\right) e^{-\frac{b}{m}t}}$$

If the initial velocity is zero, the equation simplifies

$$v_y(t) = v_{ter}\left(1 - e^{-\frac{b}{m}t}\right)$$



One can then find y as a function of time by integrating the equation for v_y

$$y(t) = \int_0^t v_y(t)\,dt + y_0$$

rem $\tau = \frac{m}{b}$

$$= \int_0^t \left[v_{ter} + (v_{y,0} - v_{ter})\, e^{-\frac{t}{\tau}} \right] dt + y_0$$

$$= v_{ter}\, t + (v_{0,y} - v_{ter}) \left[-\tau\, e^{-\frac{t}{\tau}} \right]_0^t + y_0$$

$$\boxed{y(t) = v_{ter}\, t + (v_{0,y} - v_{ter})\, \tau \left(1 - e^{-\frac{t}{\tau}} \right) + y_0}$$