Quadratic drag: Vertical motion

Monday, June 24, 2019 12:00 PM

One can also integrate the equation of motion for an object in vertical motion (near the surface of the earth) and subject to a quadratic drag force. By taking the y axis pointing downward one can write

$$m\dot{v} = mg - cv^2$$

The weight and the drag force will cancel out at the terminal velocity

$$c\,v_{ter}^2 = mg \longrightarrow v_{ter} = \sqrt{\frac{mg}{c}}$$

The equation of motion becomes then

$$\frac{dv}{dt} = g - \frac{c}{m}v^2 = g\left(1 - \frac{v^2}{v_{ter}^2}\right)$$

Using the separation of variables one finds

$$\frac{dv}{1 - \frac{v^2}{v_{ter}^2}} = g\,dt \qquad u \equiv \frac{v}{v_{ter}}$$

$$v_{ter}\int_0^{\frac{v}{v_{ter}}} \frac{du}{1-u^2} = gt \longrightarrow \frac{v_{ter}}{g}\operatorname{arctanh}(u)\Big|_0^{\frac{v}{v_{ter}}} = t$$

$$\operatorname{arctanh}\left(\frac{v}{v_{ter}}\right) = \frac{gt}{v_{ter}}$$

$$\frac{v}{v_{ter}} = \tanh \frac{gt}{v_{ter}} \longrightarrow \boxed{v = v_{ter} \tanh \frac{gt}{v_{ter}}}$$

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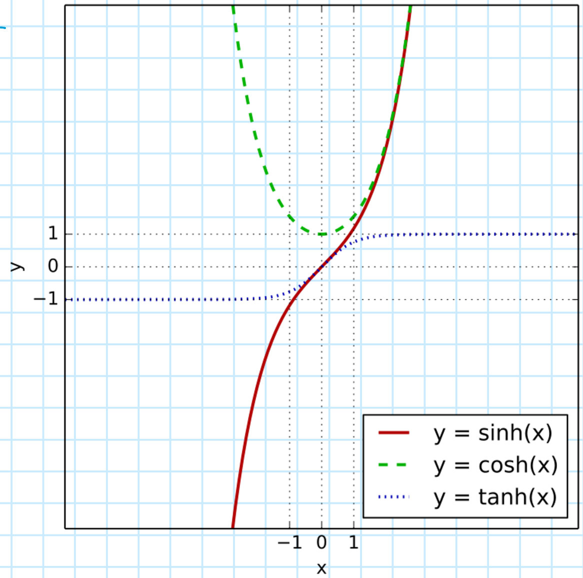

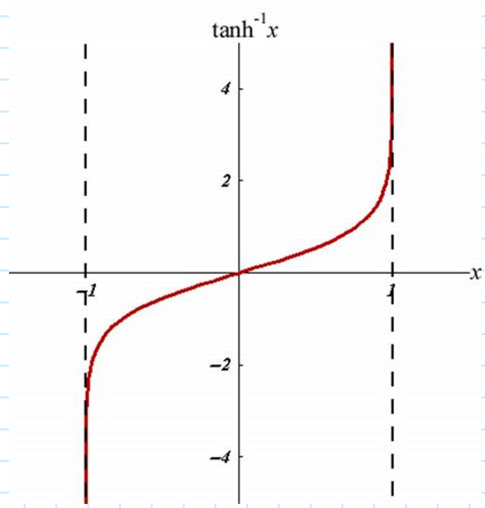

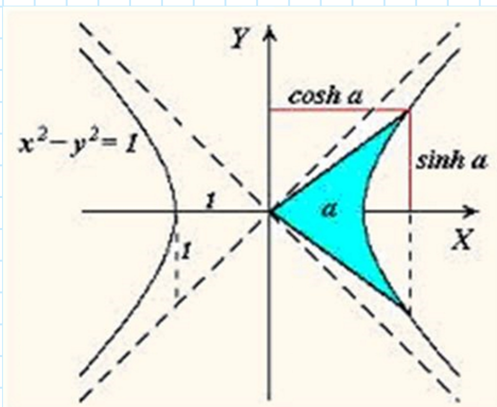

One can now integrate the equation for v in order to find y

$$\frac{dy}{dt} = v_{ter} \tanh\left(\frac{gt}{v_{ter}}\right)$$

$$\int_{y_0}^{y} dy' = v_{ter} \int_0^t \tanh\left(\frac{gt'}{v_{ter}}\right) dt'$$

$$y - y_0 = \frac{v_{ter}^2}{g} \int_0^{\frac{gt}{v_{ter}}} \tanh u \;\; du$$

$$\boxed{y = y_0 + \frac{v_{ter}^2}{g} \ln\left[\cosh\left(\frac{gt}{v_{ter}}\right)\right]}$$