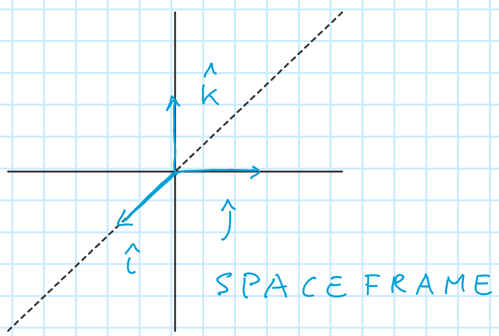
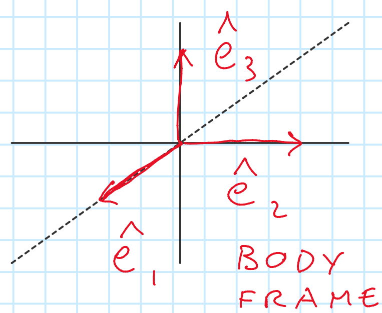
Euler angles

Wednesday, December 11, 2019 7:49 AM

Euler equations are written with respect to the principal axes, which are fixed axes in the rotating body. This makes them difficult to deal with in general situations. One needs to find equations that refer to non rotating frames. In order to do so one needs to find a convenient way to express the orientation of a rigid body with respect to a fixed, non rotating frame of reference. One way of doing this is to use Euler's angles.

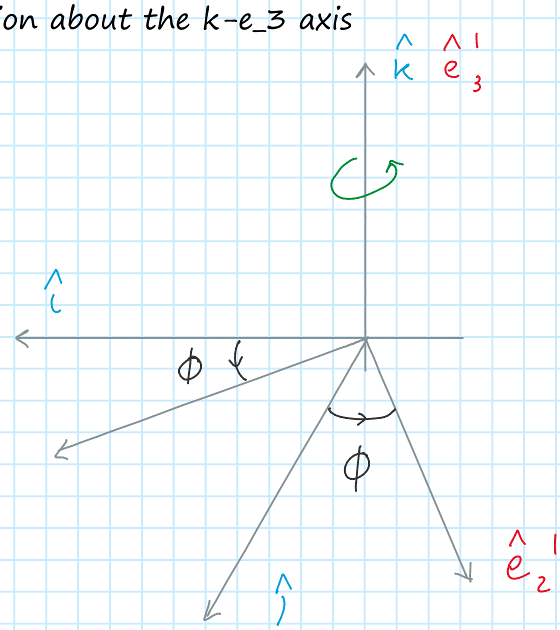
We restrict our discussion to the case of an object rotating about a fixed point and we choose the fixed point as the origin of both the space and then body frame. One can describe a generic orientation of a rigid body in three steps as shown below.



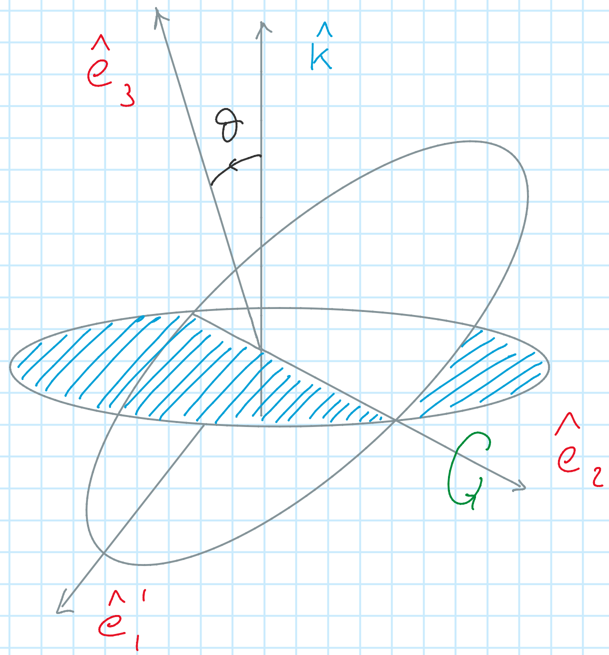


Begin from a situation in which the two frames coincide, with k and e_3 in the same direction, and j and e_2 pointing in the same direction.

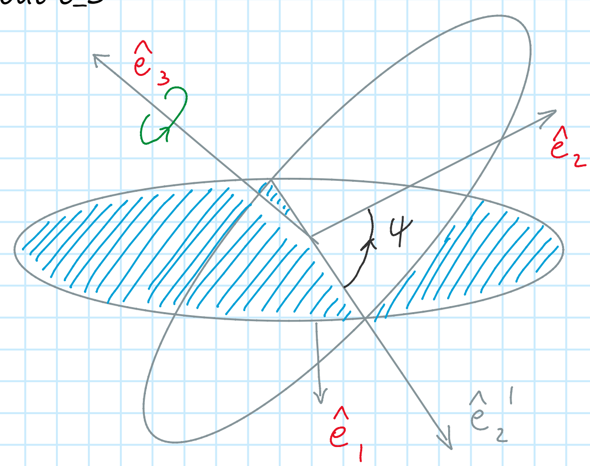
Step 1: Rotation about the k-e_3 axis



Step 2: Rotation about the e_2 prime axis



Step 3: Rotation about e_3



The angles Φ θ and ψ are called Euler's angles. They specify the rigid body orientation with respect to the fixed space frame.

At this point one can write the angular velocity easily, if one is willing to deal with a mix of unit vectors belonging to different frames:

$$\bar{\omega} = \dot{\phi}\,\hat{k} + \dot{\vartheta}\,\hat{e}_2' + \dot{\psi}\,\hat{e}_3$$

One can then rewrite all of the unit vectors either in terms of the unit vectors of either the body frame or of the space frame:

$$\hat{e}_2' = -\sin\phi \, \hat{\imath} + \cos\phi \, \hat{\jmath}$$

$$\hat{e}_3 = \sin\vartheta\cos\phi \, \hat{\imath} + \sin\vartheta\sin\phi \, \hat{\jmath} + \cos\vartheta \, \hat{k}$$

therefore

$$\begin{aligned}\bar{\omega} = & \left(\dot{\psi}\sin\vartheta\cos\phi - \dot{\vartheta}\sin\phi\right)\hat{\imath} \\ & + \left(\dot{\vartheta}\cos\phi + \dot{\psi}\sin\vartheta\sin\phi\right)\hat{\jmath} \\ & + \left(\dot{\phi} + \dot{\psi}\cos\vartheta\right)\hat{k}\end{aligned}$$

Similarly, if one wants to obtain the angular velocity in the body frame, one needs to make use of the relations

$$\begin{aligned}\hat{e}_2' &= \sin\psi \, \hat{e}_1 + \cos\psi \, \hat{e}_2 \\ \hat{k} &= -\sin\vartheta \, \hat{e}_1' + \cos\vartheta \, \hat{e}_3 \\ &= -\sin\vartheta\cos\psi \, \hat{e}_1 + \sin\vartheta\sin\psi \, \hat{e}_2 + \cos\vartheta \, \hat{e}_3\end{aligned}$$

$$\begin{aligned}\bar{\omega} = & \left(-\dot{\phi}\sin\vartheta\cos\psi + \dot{\vartheta}\sin\psi\right)\hat{e}_1 \\ & + \left(\dot{\phi}\sin\vartheta\sin\psi + \dot{\vartheta}\cos\psi\right)\hat{e}_2 \\ & + \left(\dot{\phi}\cos\vartheta + \dot{\psi}\right)\hat{e}_3\end{aligned}$$

Notice that when we deal with an object that is symmetric with respect to rotations around e_3, two of the principal moments are identical and one can choose any pair of axes perpendicular to e_3 as principal axes. In such a case one can simply choose e_2 prime as a principal axis. In that case the angular velocity with respect to the principal axes is

$$\bar{\omega} = \left(-\dot{\phi}\sin\vartheta\right)\hat{e}_1' + \dot{\vartheta}\,\hat{e}_2' + \left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)\hat{e}_3$$

Consequently the angular momentum will be

$$\bar{L} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \bar{\omega}$$

rem $\lambda_2 = \lambda_1$
because of symm.

$$= \left(-\lambda_1 \dot{\phi} \sin\vartheta\right) \hat{e}_1' + \lambda_1 \dot{\vartheta}\, \hat{e}_2' + \lambda_3 \left(\dot{\psi} + \dot{\phi}\cos\vartheta\right) \hat{e}_3$$

It is sometimes useful to have the third component of the angular momentum in the space frame rather than in the body frame:

$$\hat{e}_1' = \cos\vartheta \cos\phi\, \hat{\imath} + \cos\vartheta \sin\phi\, \hat{\jmath} - \sin\vartheta\, \hat{k}$$

$$\hat{e}_2' = -\sin\phi\, \hat{\imath} + \cos\phi\, \hat{\jmath}$$

$$\hat{e}_3 = \sin\vartheta \cos\phi\, \hat{\imath} + \sin\vartheta \sin\phi\, \hat{\jmath} + \cos\vartheta\, \hat{k}$$

$$L_z = \lambda_1 \dot{\phi} \sin^2\vartheta + \lambda_3 \left(\dot{\psi} + \dot{\phi}\cos\vartheta\right) \cos\vartheta$$

$$= \lambda_1 \dot{\phi} \sin^2\vartheta + L_3 \cos\vartheta$$

From the equation above one can extract the useful relation

$$\dot{\phi} = \frac{L_z - L_3 \cos\vartheta}{\lambda_1 \sin^2\vartheta}$$

Finally, the kinetic rotational energy of the object can be written as

$$T = \frac{1}{2} \lambda_1 \Big(\underbrace{\dot{\phi}^2 \sin^2\vartheta}_{\omega_1^2} + \underbrace{\dot{\vartheta}^2}_{\omega_2^2}\Big) + \frac{1}{2} \lambda_3 \underbrace{\left(\dot{\psi} + \dot{\phi}\cos\vartheta\right)^2}_{\omega_3^2}$$