Euler equations with zero torque

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Euler equations in the case of zero torque become

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\,\omega_2 \omega_3$$
$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\,\omega_3 \omega_1$$
$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\omega_1 \omega_2$$

Three different principal moments

One can start by considering the general case of a body with three different principal moments.
Let's assume that at the time t=0 the body spins around the third principal axis, so that

$$\overline{\omega} = \omega_3 \hat{e}_3 \,, \qquad \omega_1 = \omega_2 = 0 \quad @\ t=0$$

The equations become

$$\lambda_1 \dot{\omega}_1 = \lambda_2 \dot{\omega}_2 = \lambda_3 \dot{\omega}_3 = 0$$

Therefore the angular velocity in the rotating body frame remains constant.
However, the angular momentum will then be

$$\overline{L} = \lambda_3 \overline{\omega}$$

In addition we know that, since there is no torque acting on the object, the angular momentum is constant as seen in any inertial frame. One can then conclude that

If a body that is subject to no torque is spinning initially about any principal axis, it will continue to do so indefinitely with constant angular velocity.

Notice that if at a given time the angular velocity is NOT parallel to a principal axis, then then vector angular velocity is not constant. This can be seen by looking at Euler equations and observing that if two of the components of ω are not zero, one of the time derivatives of the components of ω are not zero. For example

$$\omega_3 \neq 0 \quad \wedge \quad \omega_2 \neq 0 \qquad\qquad \lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_1)\,\omega_2 \omega_3 \neq 0$$

One can then conclude that the only way a body with three different principal moments can rotate freely with constant angular velocity is by rotating about one of its principal axes.

It is interesting to ask if this free rotation is stable. In other words if one provides a small angular velocity to the object along the other two principal axes, are these small angular velocities going to grow or are they going to die out?
In order to answer this question imagine that an object is rotating along the principal axis e_3 with a velocity ω_3. Let's then give an infinitesimal angular velocity to the object along the other two axes

$$\lambda_3 \dot{\omega}_3 = (\lambda_1 - \lambda_2)\,\underbrace{\omega_1 \omega_2}_{\text{infinitesimal squared}} \simeq 0$$

The angular velocity along the third principal axis remains constant. The other two equations are

$$\lambda_1 \dot{\omega}_1 = (\lambda_2 - \lambda_3)\,\omega_3 \omega_2$$

$$\lambda_2 \dot{\omega}_2 = (\lambda_3 - \lambda_1)\,\omega_3 \omega_1$$

$$\lambda_1 \ddot{\omega}_1 = (\lambda_2 - \lambda_3)\,(\omega_3 \dot{\omega}_2 + \cancelto{0}{\dot{\omega}_3}\,\omega_2)$$

$$= (\lambda_2 - \lambda_3)\,\omega_3 \frac{\omega_3}{\lambda_2}(\lambda_3 - \lambda_1)\,\omega_1$$

$$\ddot{\omega}_1 = -\left[\frac{(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2)}{\lambda_1 \lambda_2}\,\omega_3^2\right]\omega_1$$

If the coefficient in square brackets is positive, the solution of the equation is a sine or a cosine, the angular velocity along the first and the second principal axes remains small. Notice in fact that the time derivative of ω_2 is proportional to ω_ 1. The coefficient in square brackets is positive if

$$\lambda_3 > \lambda_1 \quad \wedge \quad \lambda_3 > \lambda_2 \qquad \text{or} \qquad \lambda_3 < \lambda_1 \quad \wedge \quad \lambda_3 < \lambda_2$$

If instead

$$\lambda_1 < \lambda_3 < \lambda_2 \qquad \text{or} \qquad \lambda_2 < \lambda_3 < \lambda_1$$

The coefficient in square brackets and the solution of a differential equation is an exponential and the corresponding component of the angular velocity would grow fast. For a freely spinning body, rotations about the principal axis with intermediate moment of inertia are unstable.

Motion of a body with two equal moments of inertia

Euler equations become easier to solve if two of the principal moments are equal. This case is physically relevant because it correspond for example to the case of a spinning top. Keeping the assumption of zero external torque, let' assume that

$$\lambda_1 = \lambda_2$$

One can immediately conclude that

$$\dot{\omega}_3 = 0 \longrightarrow \omega_3 = \text{const.}$$

The other two equations become

$$\dot{\omega}_1 = \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \omega_2 \equiv \Omega_b \, \omega_2$$

$$\dot{\omega}_2 = - \frac{\lambda_1 - \lambda_3}{\lambda_1} \omega_3 \omega_1 = - \Omega_b \, \omega_1$$

Now one can introduce $\eta = \omega_1 + i \, \omega_2$

$$\dot{\eta} = -i\Omega_b \eta \longrightarrow \eta = \eta_0 e^{-i\Omega_b t}$$

One can then choose the axes in such a way that

@ $t=0$ $\quad \omega_1 \equiv \omega_0 \quad \omega_2 = 0 \quad \eta_0 = \eta(t=0) = \omega_0$

$$\bar{\omega} = \omega_0 \cos(\Omega_b t)\hat{e}_1 - \omega_0 \sin(\Omega_b t)\hat{e}_2 + \omega_3 \hat{e}_3$$

As seen from the body, the angular velocity moves in a cone around the e_3 axis. The cone is called the body cone and the angular velocity for this rotation is Ω_b. The angular momentum is

$$\bar{L} = \lambda_1 \omega_0 \cos(\Omega_b t)\hat{e}_1 - \lambda_1 \omega_0 \sin(\Omega_b t)\hat{e}_2 + \lambda_3 \omega_3 \hat{e}_3$$

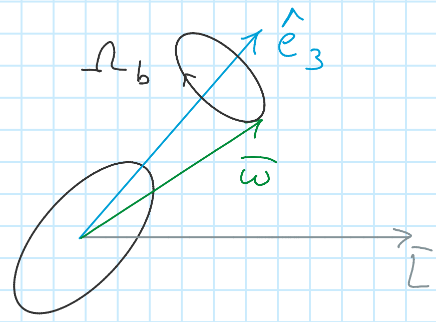
$\bar{L}$, $\bar{\omega}$, and $\hat{e}_3$ are coplanar

Both the angular velocity and the angular momentum precess around the third principal axis with angular speed Ω_b.

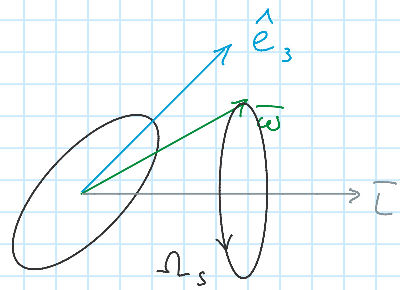
In an inertial frame L is constant. Consequently in that frame the angular velocity and the third principal axis precess about L. In particular ω describes a cone in the space frame. This cone is called the space cone. It can be shown that the rate of precession of the angular velocity along L is

$$\Omega_s = \frac{L}{\lambda_1}$$

Notice that these precession motions are not due to the presence of a torque. We assumed that the net torque on the object is zero at the beginning of this discussion.
The situation is better understood with a figure



BODY FRAME



SPACE FRAME