Rotational kinetic energy

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The rotational kinetic energy of an object can be written as

$$T = \frac{1}{2}\, \bar{\omega} \cdot \bar{L}$$

Proof

The total kinetic energy of an object that is only spinning with angular velocity ω can be written as

$$T = \sum_{\alpha=1}^{N} \frac{1}{2} m_\alpha v_\alpha^2 = \sum \frac{1}{2} m_\alpha \bar{v}_\alpha \cdot \bar{v}_\alpha$$

$$= \sum_{\alpha} \frac{1}{2} m_\alpha (\bar{\omega} \times \bar{r}_\alpha) \cdot (\bar{\omega} \times \bar{r}_\alpha)$$

Now we can use the identity

$$(\bar{a} \times \bar{b})^2 = \varepsilon_{ijk} a_j b_k \; \varepsilon_{ilm} a_l b_m$$

$$= \varepsilon_{ijk} \varepsilon_{ilm} \; a_j b_k a_l b_m$$

$$= (\delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}) a_j a_l b_k b_m$$

$$= a^2 b^2 - (\bar{a} \cdot \bar{b})^2$$

$$(\bar{\omega} \times \bar{r}_\alpha)^2 = \omega^2 r_\alpha^2 - (\bar{\omega} \cdot \bar{r}_\alpha)^2$$

Therefore

$$T = \frac{1}{2} \sum_{\alpha=1}^{N} m_\alpha \left[(\omega r_\alpha)^2 - (\bar{\omega} \cdot \bar{r}_\alpha)^2 \right]$$

Now let's consider the total angular momentum of the object

$$\bar{L} = \sum_{\alpha=1}^{N} \bar{\ell}_\alpha = \sum_\alpha \bar{r}_\alpha \times m_\alpha \bar{v}_\alpha = \sum_\alpha m_\alpha \bar{r}_\alpha \times (\underbrace{\bar{\omega} \times \bar{r}_\alpha}_{\bar{v}_\alpha})$$

$$[\bar{r} \times (\bar{\omega} \times \bar{r})]_i = \varepsilon_{ijk}\, r_j \left(\varepsilon_{k\ell m}\, \omega_\ell\, r_m \right)$$

$$= \varepsilon_{kij}\, \varepsilon_{k\ell m}\, r_j\, \omega_\ell\, r_m$$

$$= \left(\delta_{i\ell}\, \delta_{jm} - \delta_{im}\, \delta_{j\ell} \right) r_j\, \omega_\ell\, r_m$$

$$= \omega_i\, r^2 - r_i\, \bar{\omega} \cdot \bar{r}$$

$$\bar{r}_\alpha \times (\bar{\omega} \times \bar{r}_\alpha) = \bar{\omega}\, r_\alpha^2 - \bar{r}_\alpha\, \bar{\omega} \cdot \bar{r}_\alpha$$

$$\bar{L} = \sum_\alpha m_\alpha \left[\bar{\omega}\, r_\alpha^2 - \bar{r}_\alpha\, \bar{\omega} \cdot \bar{r}_\alpha \right]$$

$$\bar{\omega} \cdot \bar{L} = \underbrace{\sum_\alpha m_\alpha \left[\omega^2 r_\alpha^2 - (\bar{\omega} \cdot \bar{r}_\alpha)^2 \right]}_{2T}$$

$$\boxed{T = \frac{1}{2}\, \bar{\omega} \cdot \bar{L}}$$

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In addition, since

$$\bar{L} = I \cdot \bar{\omega} \longrightarrow T = \frac{1}{2}\, \tilde{\bar{\omega}} \cdot I \cdot \bar{\omega} = \frac{1}{2}\, \underbrace{\omega_i\, I_{ij}\, \omega_j}_{\substack{\text{repeated}\\ \text{indices are}\\ \text{summed}\\ \text{over}}}$$

~ indicates the transpose

Finally, if the frame of reference is aligned to the principal axes so that then tensor of inertia is diagonal

$$T = \frac{1}{2} \omega_i I_{ij} \omega_j = \frac{1}{2} (\omega_1, \omega_2, \omega_3) \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$= \frac{1}{2} \left(\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2 \right)$$