Principal axes of inertia

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Whenever the angular velocity of an object is pointing in the same direction as the angular momentum we say that that particular direction is a principal axis of the body with respect to the chosen origin of the reference frame.

- Choose a frame
- if $\bar{L} = \lambda \bar{\omega}$ $\lambda \in \mathbb{R}$ $\longrightarrow$ $\frac{\bar{\omega}}{\omega}$ is the direction of a principal axis

It is easy to see that λ is the moment of inertia about the principal axis

assume that $\bar{\omega} = \omega \hat{k}$ and $\bar{L} = \lambda \bar{\omega}$

It must still be true that $\bar{L} = I \cdot \bar{\omega}$

$$\bar{L} = I \cdot \bar{\omega} = I_{xz}\,\omega\,\hat{i} + I_{yz}\,\omega\,\hat{j} + I_{zz}\,\omega\,\hat{k} = \lambda \hat{k}$$

$\hookrightarrow$ $I_{xz} = I_{yz} = 0$ $I_{zz} = \lambda$ PRINCIPAL MOMENT

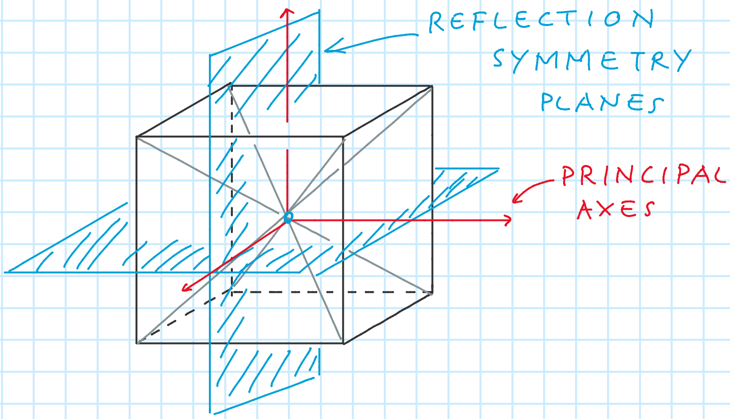
With a similar reasoning it is possible to see that if all of the three axes are principal axes the inertia tensor is diagonal

$$I = \begin{pmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{pmatrix}$$

This is indeed what happened in the example of the inertia tensor of the cone calculated with respect to its tip.

In general, if the body in question has a rotational symmetry axis going through the origin of the frame of reference, that axis is a principal axis. In addition, any two axes perpendicular to the symmetry axis are also principal axes.

If an object has a reflection symmetry with respect to two perpendicular planes, like it happens for a cube, then the three axes defined by these planes are principal axes.



However, the existence of principal axes is not related to the symmetry of the object, but to the diagonazability of the inertia tensor

For any rigid body and any point O there are three perpendicular principal axes through O. The inertia tensor with respect to those axes is diagonal. If the object rotates about those axes the angular momentum associated to that rotation is parallel to the angular velocity.

In mathematical term, the inertia tensor, which is a symmetric 3x3 matrix, is diagonalizable.

The discussion of the rotational motion is in general simplified by the use of the principal axes as the reference frame employed in a given problem. However, the principal axes are fixed with respect to the rigid body. This means that in general they will rotate and translate in space.