Liouville theorem

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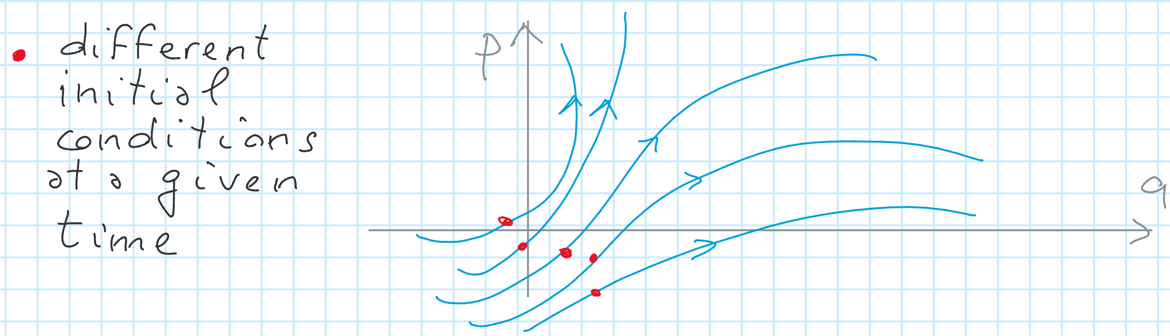
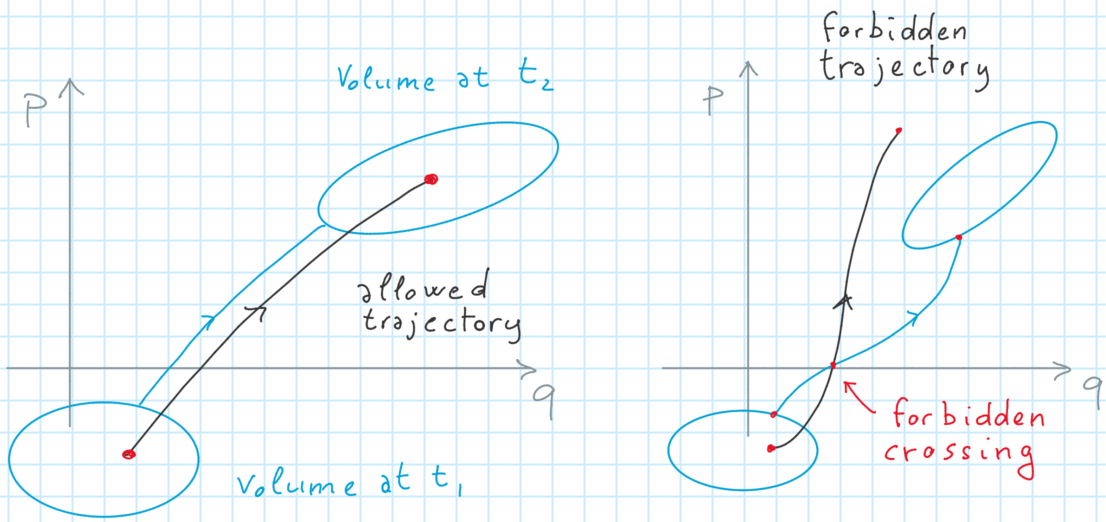
We already observed that one can follow the motion of a system as the motion of a dot in phase space that represents that system. Given an initial condition in phase space, the system will follow a trajectory in phase space that will never intersect a trajectory starting from another initial condition which is not in the future or in the past of the first trajectory. One can say that the phase space is filled by infinitely many thin spaghetti, that never intersect each other. The simplest example is of course a two dimensional phase space.



In some cases, such as in statistical mechanics, it is interesting to follow the motion of many copies of the system in phase space. These copies will differ because of the initial conditions and will follow different trajectories. For example, when one studies an ideal gas, the copies of the system are the molecules, each one of which satisfies the same Hamiltonian but has a different initial velocity and position.
Since each molecule will have generalized coordinates and momenta that define its position in phase space at any given time, one can study the motion of a swarm or these points in phase space as time goes by. The cloud of points can change in shape as the system evolves in time.
It is often useful to visualize the simple case of a two dimensional phase space in which the phase space point and the corresponding phase space velocity are

$$\bar{z} = \{q, p\} \qquad \dot{\bar{z}} = \{\dot{q}, \dot{p}\} = \left\{\frac{\partial \mathcal{H}}{\partial p}, -\frac{\partial \mathcal{H}}{\partial q}\right\}$$

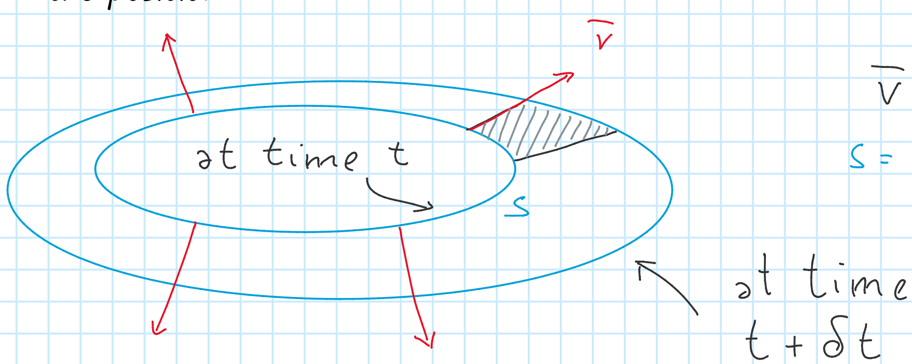
One can observe that if one considers the trajectories starting from a certain closed volume in phase space, these trajectories will end up after a given time in a different closed volume of phase space. No point that starts from inside the initial volume will end up outside the final volume. Indeed if that could happen, it would mean that the trajectory of a point that started from inside the volume crossed a trajectory of a point of the initial volume boundary. This is forbidden because trajectories cannot cross.



Liouville theorem, that we want to prove, states that the volume of the phase space that we consider does not change in time. In order to prove the theorem, one needs to calculate the rate of change of a volume in terms of the velocity of the points in the volume, and to use the div divergence theorem (Gauss' theorem).

Rate of change of a volume

One can calculate the rate of change of a volume in three dimensional space (think about a volume of gas expanding or contracting in a balloon). The volume is characterized by a bunch of points occupying a given position in space at a given time. Each of these points has a velocity that can (and in general will) depend on the position



$$\bar{v} = \bar{v}(\bar{r})$$

S = Surface of the volume

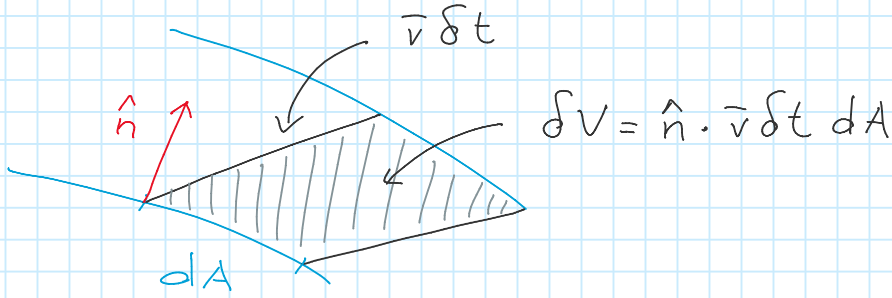
This is equivalent to the case of a phase space point, where the velocity also depends on the position.

$$\bar{z} = \{\bar{q}, \bar{p}\} \qquad \dot{\bar{z}} = \{\dot{\bar{q}}, \dot{\bar{p}}\}$$

phase space point — phase space point velocity

Consequently the result of the three dimensional calculation discussed below is applicable to the case of a phase space volume as well.

The infinitesimal volume of the gray portion in the previous figure is



Therefore the total change in volume in the time δt is

$$\delta V = \int_S \hat{n}\cdot\bar{v}\,\delta t\, dA$$

Consequently, since also δt is an infinitesimal

$$\frac{dV}{dt} = \int \hat{n}\cdot\bar{v}\, dA$$

Notice that

if $\hat{n}\cdot\bar{v} > 0 \longrightarrow$ V is locally expanding

if $\hat{n}\cdot\bar{v} < 0 \longrightarrow$ V is locally contracting

One can then combine the above with the divergence theorem to find

$$\frac{dV}{dt} = \int_S \hat{n}\cdot\bar{v}\, dA = \int_V \nabla\cdot\bar{v}\, dV$$

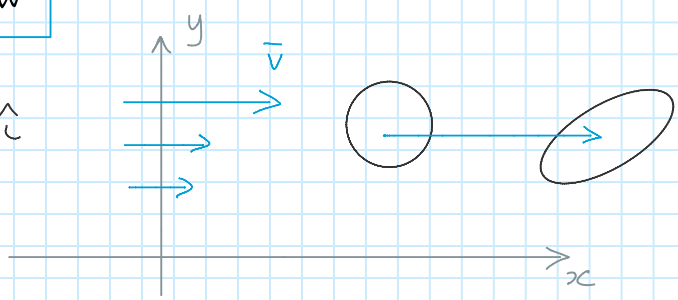
rem

$$\nabla\cdot\bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}$$

The relation above has the important consequence that if the divergence of the velocity field is zero, then the volume is invariant. That is precisely the situation of a fluid in laminar flow.

Laminar flow

$$\bar{v} = k y \hat{i}$$



Consider the case of a fluid that moves as collection of rigid flat surfaces stacked up along the y axis. Let's assume that the speed of the layers grows linearly with y and the fluid moves along x. It is easy to see that

$$\nabla \cdot \bar{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} = \frac{\partial}{\partial x}(ky) = 0$$

Consequently, the volume of a sphere of fluid would remain constant as the fluid moves, even if the sphere will be distorted in another shape.

$$\frac{dV}{dt} = \int_V \nabla \cdot \bar{v} \, dv = 0$$

Since we need to apply the theorem to phase space rather than to ordinary three dimensional space, we need to consider the generalization of the divergence to n dimensions, which is straightforward

$$\nabla \cdot \bar{v} = \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \ldots + \frac{\partial v_n}{\partial x_n}$$

Liouville's theorem

The proof of Liouville's theorem is now straightforward. For a two dimensional phase space, the velocity of a phase space point is divergenceless

$$\bar{v} = \dot{\bar{z}} = \{\dot{q}, \dot{p}\}$$

$$\nabla \cdot \bar{v} = \frac{\partial \dot{q}}{\partial q} + \frac{\partial \dot{p}}{\partial p} = \frac{\partial}{\partial q}\left(\frac{\partial \mathcal{H}}{\partial p}\right) + \frac{\partial}{\partial p}\left(-\frac{\partial \mathcal{H}}{\partial q}\right) = 0$$

Consequently

$$\frac{dV}{dt} = \int_V \nabla \cdot \bar{v} \, dV = 0$$

It simple to generalize the result to a phase space of dimension 2n since

$$\bar{v} = \dot{\bar{z}} = \{\dot{q}_i, \dot{p}_i\} \qquad i = 1, \dots, n$$

$$\nabla \cdot \bar{v} = \sum_{i=1}^{n}\left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i}\right) = \sum_{i=1}^{n}\left(\frac{\partial \mathcal{H}}{\partial q_i \partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i \partial q_i}\right) = 0$$

Liouville's theorem remains valid for Hamiltonian that explicitly depend on time and for systems that cannot be described in terms of natural coordinates (so that the Hamiltonian does not coincide with the energy). Notice that there is no analog of Liouville's theorem in the space of generalized positions and velocities, which is the natural space to use in Lagrangian mechanics.

Finally Liouville' theorem has important consequences in the study of chaotic systems, where phase space trajectories that are close at one point in time diverge rapidly from one another.