Hamilton equations in several dimensions

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It is now necessary to generalize the derivation of Hamilton's equations to the case of systems depending on several generalized coordinates. For this purpose, it is useful to group the generalized coordinates and the generalized moments in n dimensional vectors, where n is the number of generalized coordinates.

$$\bar{q} = (q_1, q_2, \ldots, q_n) \qquad \bar{p} = (p_1, p_2, \ldots, p_n)$$

The goal here is to prove that Hamilton's equations are a consequence of the fact that a given system satisfies Lagrange's equations. For this purpose we assume

1) The physical system under study satisfies Lagrange equations
2) The constraints if present are holonomic, i.e. the number of degrees of freedom is identical to the number of generalized coordinates.
3) Forces which are not constraining forces are described by a potential, but the forces can be non conservative, i.e. the potential can explicitly depend in time.
4) The equations that link the Cartesian coordinates of the various points in the system to the generalized coordinates can explicitly depend on time, i.e. one does not need to require to be dealing with natural generalized coordinates.

With these assumptions, one can write the Lagrangian and Lagrange equations

$$\mathcal{L} = \mathcal{L}(\bar{q}, \dot{\bar{q}}, t) = T - U$$

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \qquad i = 1, \ldots, n$$

One can then define the Hamiltonian and the generalized momenta as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

$$p_i = \frac{\partial \mathcal{L}(\bar{q}, \dot{\bar{q}}, t)}{\partial \dot{q}_i} \qquad [i = 1, 2, \ldots, n]$$

By inverting the relations defining the generalized momenta one can then write the generalized velocities as functions of q and p s.

$$\dot{q}_i = \dot{q}_i(q_1, q_2, \dots, q_n, p_1, p_2, \dots, p_n, t) \qquad [i = 1,\dots n]$$

$$\dot{\bar{q}} = \dot{\bar{q}}(\bar{q}, \bar{p}, t)$$

Therefore the Hamiltonian will become

$$\mathcal{H}(\bar{q}, \bar{p}, t) = \sum_{i=1}^{n} p_i \dot{q}_i(\bar{q}, \bar{p}, t) - \mathcal{L}(\bar{q}, \dot{\bar{q}}(\bar{q}, \bar{p}, t), t)$$

Subsequently one can proceed as in the one dimensional case. One can start by taking the derivative of the Hamiltonian with respect to p

$$\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i + \sum_{j=1}^{n} \left(p_j \frac{\partial \dot{q}_j}{\partial p_i} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial p_i} \right) = \dot{q}_i$$

One can then take the derivative of H with respect to q

$$\frac{\partial \mathcal{H}}{\partial q_i} = \sum_{j=1}^{n} \left(p_j \frac{\partial \dot{q}_j}{\partial q_i} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_j}}_{p_j} \frac{\partial \dot{q}_j}{\partial q_i} \right) - \frac{\partial \mathcal{L}}{\partial q_i}$$

$$= -\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}_i} = -\dot{p}_i$$

$$\boxed{\frac{\partial \mathcal{H}}{\partial p_i} = \dot{q}_i \qquad \frac{\partial \mathcal{H}}{\partial q_i} = -\dot{p}_i}$$

HAMILTON EQUATIONS
2n equations

Time derivative of the Hamiltonian

One can then consider the total time derivative of the Hamiltonian.

$$\frac{d\mathcal{H}}{dt} = \sum_{i=1}^{n}\left(\underbrace{\frac{\partial \mathcal{H}}{\partial q_i}}_{= -\dot{p}_i}\dot{q}_i + \underbrace{\frac{\partial \mathcal{H}}{\partial p_i}}_{= \dot{q}_i}\dot{p}_i\right) + \frac{\partial \mathcal{H}}{\partial t}$$

because of Hamilton's equations

$$\boxed{\frac{d\mathcal{H}}{dt} = \frac{\partial \mathcal{H}}{\partial t}}$$

One can then conclude that if the Hamiltonian does not explicitly depend on time, then the total derivative of the Hamiltonian with respect to time is zero, and the Hamiltonian is a conserved quantity. In addition, we already know that if we deal a system in which the generalized coordinates are natural, the Hamiltonian is the total energy of the system, H = T + U. If an Hamiltonian written in terms of natural coordinates does not explicitly depend on time, then the total energy of the system is a conserved quantity.