Hamiltonian equations - one dimension

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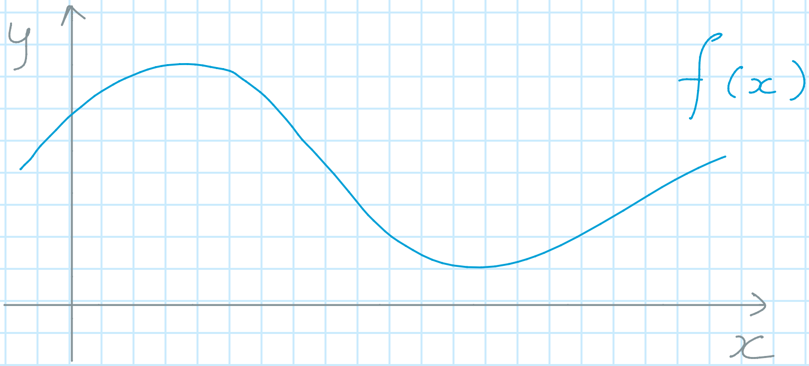
In order to introduce Hamilton equations we consider a system that depends on a single generalized and natural coordinate, q. In that case the Lagrangian has the following form

$$\mathcal{L}(q,\dot{q}) = T(q,\dot{q}) - U(q)$$

Since we are dealing with conservative forces, such that it is possible to write a potential, that potential can only depend on the configuration of the system and therefore on the coordinate q, but not on its derivative.

On the contrary the kinetic energy can depend on q in addition to the corresponding generalized velocity (the time derivative of q). Consider for example the case of a bead that can move without friction along a wire described by a function y = f(x)



$$ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

$$dx\sqrt{1+(f'(x))^2}$$

$$v = \frac{ds}{dt} = \dot{x}\sqrt{1+(f'(x))^2}$$

$$\boxed{T(x,\dot{x}) = \frac{1}{2}m\left[1+(f'(x))^2\right]\dot{x}^2}$$

$$\boxed{\mathcal{L} = \frac{1}{2} m \left[1 + \left(f'(x) \right)^2 \right] \dot{x}^2 - m g f(x)}$$

One can therefore see that the Lagrangian has in general the functional form

$$\mathcal{L}(q, \dot{q}) = \frac{1}{2} A(q) \dot{q}^2 - U(q)$$

Of course it is easy to find examples of Lagrangians where the kinetic energy does not depend on q, for example the simple pendulum, where the generalized coordinate q is the angle that the pendulum makes with the vertical direction

$$\mathcal{L}(\vartheta, \dot{\vartheta}) = \frac{1}{2} m \ell^2 \dot{\vartheta}^2 - m g \ell (1 - \cos\vartheta)$$

↳ pendulum length

However we want to reason in terms of the most general one dimensional case described by the Lagrangian in which the kinetic energy can depend on the generalized coordinate q.

The Hamiltonian is defined as

$$\mathcal{H} = p \dot{q} - \mathcal{L}(q, \dot{q})$$

Where

$$p \equiv \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q) \dot{q}$$

GENERALIZED MOMENTUM

Therefore

$$\begin{aligned} \mathcal{H} &= p \dot{q} - \mathcal{L}(q, \dot{q}) \\ &= A(q) \dot{q}^2 - \frac{1}{2} A(q) \dot{q}^2 + U(q) \\ &= \frac{1}{2} A(q) \dot{q}^2 + U(q) = T + U \end{aligned}$$

However, the Hamiltonian is a function of q and p. In order to stress this fact one can write

$$\dot{q} = \frac{p}{A(q)} = \dot{q}(q,p)$$

$$\mathcal{H}(q,p) = p\,\dot{q}(q,p) - \mathcal{L}(q,\dot{q}(q,p))$$

The next step is to find Hamilton's equations. Take the derivative of H with respect to q

$$\frac{\partial \mathcal{H}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \left[\frac{\partial \mathcal{L}}{\partial q} + \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{=\,p}\frac{\partial \dot{q}}{\partial q}\right] = -\frac{\partial \mathcal{L}}{\partial q} = -\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}} = -\dot{p}$$

Similarly

$$\frac{\partial \mathcal{H}}{\partial p} = \dot{q}(q,p) + p\frac{\partial \dot{q}}{\partial p} - \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}}}_{p}\frac{\partial \dot{q}}{\partial p} = \dot{q}$$

In summary Hamiltons equations are

$$\boxed{\frac{\partial \mathcal{H}}{\partial q} = -\dot{p} \qquad \frac{\partial \mathcal{H}}{\partial p} = \dot{q}}$$

HAMILTON'S EQUATIONS

Observe that for every generalized coordinate Hamilton's equations provide two first order differential equation. On the contrary, Lagrange equations give a single second order differential equation for every generalized coordinate.