Unbounded orbits

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One needs to analyze what happens to the orbit when the parameter $\varepsilon>1$. From the previous discussion one sees that in that case the energy is larger than zero. In addition there will be two values of the angle ϕ at which r will go to infinity

$$r(\phi) = \frac{c}{1+\epsilon \cos\phi}$$

$$r \to \infty \quad \text{if} \quad \cos\phi = -\frac{1}{\epsilon}$$

The smallest value of ε at which this can happen is 1. In that case r diverges for $\phi = \pi, -\pi$. In that case, by rewriting the equation of the orbit in cartesian coordinates one finds

$$\left.\begin{aligned} x &= r\cos\phi \\ y &= r\sin\phi \end{aligned}\right\} \longrightarrow \begin{aligned} r &= \sqrt{x^2+y^2} \\ \tan\phi &= \frac{y}{x} \end{aligned}$$

SPECIAL CASE

$$\boxed{\epsilon = 1}$$

$$r(1+\cos\phi) = c$$

$$\sqrt{x^2+y^2} + x = c$$

$$x^2+y^2 = (c-x)^2 \longrightarrow \boxed{y^2 = c^2 - 2cx}$$

PARABOLA

For an $\varepsilon>1$ one finds instead

$$r(1+\epsilon\cos\phi) = c$$

$$\sqrt{x^2+y^2} + \epsilon x = c$$

$$x^2+y^2 = (c-\epsilon x)^2 \longrightarrow x^2+y^2 = c^2+\epsilon^2 x^2 - 2c\epsilon x$$

$$(1-\epsilon^2)x^2 + 2c\epsilon x + y^2 = c^2$$

$$x^2 + \frac{2c\epsilon x}{1-\epsilon^2} + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$x^2 + \frac{2c\epsilon x}{1-\epsilon^2} + \frac{c^2\epsilon^2}{(1-\epsilon^2)^2} + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2} + \frac{c^2\epsilon^2}{(1-\epsilon^2)^2}$$

$$\left(x - \frac{c\epsilon}{\epsilon^2-1}\right)^2 + \left(\frac{y}{\sqrt{1-\epsilon^2}}\right)^2 = \frac{c^2}{(1-\epsilon^2)^2}$$

$$\delta \equiv \frac{c\epsilon}{\epsilon^2-1} \qquad \alpha \equiv \frac{c}{\epsilon^2-1} \qquad \beta \equiv \frac{c}{\sqrt{\epsilon^2-1}}$$

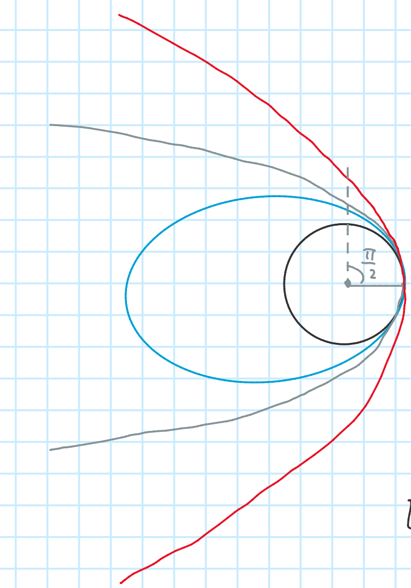
$$\boxed{\frac{(x-\delta)^2}{\alpha^2} - \frac{y^2}{\beta^2} = 1}$$

HYPERBOLA

For hyperbolas the angular range of the orbit is limited

$$-\phi_{max} \le \phi \le \phi_{max} \qquad \text{with} \qquad \cos\phi_{max} = -\frac{1}{\epsilon}$$

In summary, depending on the value of ε, one can have the following set of orbits



ECCENTRICITY	ENERGY	ORBIT
$\epsilon = 0$	$E < 0$	circle
$0 < \epsilon < 1$	$E < 0$	ellipse
$\epsilon = 1$	$E = 0$	parabola
$\epsilon > 1$	$E > 0$	hyperbola

rem

$$E = \frac{\gamma^2\mu^2}{2\ell^2}(\epsilon^2 - 1) \qquad c = \frac{\ell^2}{\gamma\mu} = \frac{\ell^2}{G m_1 m_2 \mu}$$

c = distance sun-comet when $\phi = \frac{\pi}{2}$