Non uniqueness of the Lagrangian

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The definition of the Lagrangian is not unique, in the sense that instead of defining the Lagrangian as T – U, one can turn things around and say that a Lagrangian is any function of the generalized coordinates such that Lagrange's equations provide the equations of motion for the system:

if $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ are the eqs of motion

then $\mathcal{L}$ is the Lagrangian of the system

One should observe that also with this new definition of the Lagrangian the physical path taken by the system is the one that minimizes the action integral.

In particular one should observe if one finds a Lagrangian that leads to the correct equations of motion, one can build a second equally valid Lagrangian (according to the new, more general definition of Lagrangian that we are considering here) as follows:

if $\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$ and $\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}_i} = 0$

then $\mathcal{L}' = \mathcal{L} + f$ is such that

$\frac{\partial \mathcal{L}'}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{q}_i} = 0$ are the correct eqs. of motion

It is not difficult to build a function f that satisfies Lagrange's equation, for example

$f = q_i \dot{q}_i$ (indices summed over)

$$\frac{\partial f}{\partial q_i} - \frac{d}{dt}\frac{\partial f}{\partial \dot{q}_i} = \dot{q}_i - \frac{d}{dt} q_i = 0$$

Armed with this new definition of Lagrangian one can build a Lagrangian that described the motion of a charged particle in a magnetic field.