Constrained system: Simple pendulum

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Lagrangian methods are particularly useful when one need to describe constrained systems. One of the simplest examples of a constrained system is the simple pendulum. The pendulum bob moves on a plane and its position is therefore described by two coordinates. However, these two coordinates are not independent, since the pendulum bob is always at a distance l from the pendulum fulcrum:

x^2 + y^2 = l^2

Since x and y are not independent one could thing about considering y as a function of x. That is certainly possible and prove that this system has a single degree of freedom (independent coordinate), rather than two. It is even more convenient to characterize the configuration of the pendulum by considering the angle between the vertical direction and the pendulum as the independent variable. One can then write the system's Lagrangian as a function of this angle.

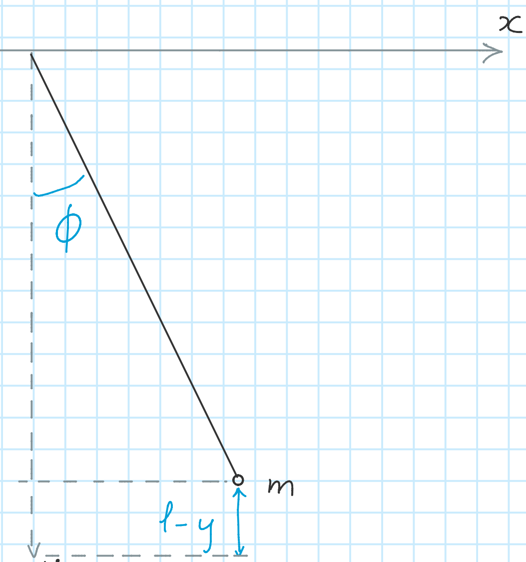
$$x = \ell \sin\phi$$

$$y = \ell \cos\phi$$

$$T = \frac{1}{2} m v^2 = \frac{1}{2} m \ell^2 \dot{\phi}^2$$

$$U = mg(\ell - y)$$

$$= mg\ell(1 - \cos\phi)$$



$$\mathcal{L} = T - U = \frac{1}{2} m \ell^2 \dot{\phi}^2 - mg\ell(1 - \cos\phi)$$

In this case one has only a generalized coordinate, l = Φ. Therefore the Lagrange equation will be

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = -mgl\sin\phi - ml^2\ddot{\phi} = 0$$

$$\boxed{\ddot{\phi} = -\frac{g}{l}\sin\phi}$$

As expected, the equation above coincides with the usual equation that one finds in Newtonian mechanics by considering the constraining force (in this case the tension). For small Φ, one recovers as expected the equation for the harmonic oscillator.

Another way to read the equation above is to recognize that

$mgl\sin\phi$ = TORQUE applied by mg on the pendulum bob, calculated w.r.t. the fulcrum

$$\frac{\partial \mathcal{L}}{\partial \phi} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = 0 \longrightarrow \frac{d}{dt}\left(\underbrace{ml^2}_{I}\underbrace{\dot{\phi}}_{\omega}\right) = \tau$$

$$\boxed{\frac{d}{dt}L = \tau}$$

"rotational" Newton's 2nd Law