Simple harmonic motion

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The equation of motion of an object of mass m that is subject to Hooke's force is

$$m\ddot{x} = -kx \longrightarrow \boxed{\ddot{x} = -\frac{k}{m}x}$$

The equation above can be rewritten as

$$\boxed{\ddot{x} = -\omega^2 x} \quad \text{with} \quad \omega = \sqrt{\frac{k}{m}} \qquad [\omega] = \left[\frac{1}{s}\right]$$

The goal is now to find general solution for this differential equation.
A first general solution (involving as expected two constants to be fixed by imposing appropriate initial conditions) is

$$\boxed{x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}}$$

EXPONENTIAL SOLUTION

Of course x should be real, while the exponentials are complex, so that the constants C_1 and C_2 should be chosen in such a way that x turns out to be real.
Of course one can rewrite the exponentials in terms of sign and cosines

$$e^{\pm i\omega t} = \cos\omega t \pm i\sin\omega t$$

$$x(t) = C_1(\cos\omega t + i\sin\omega t) + C_2(\cos\omega t - i\sin\omega t)$$

$$= \underbrace{(C_1 + C_2)}_{\equiv B_1}\cos\omega t + \underbrace{i(C_1 - C_2)}_{\equiv B_2}\sin\omega t$$

$$\boxed{x(t) = B_1\cos\omega t + B_2\sin\omega t}$$

SINUSOIDAL SOLUTION

The constants B_1 and B_2 must be real.
Observe that

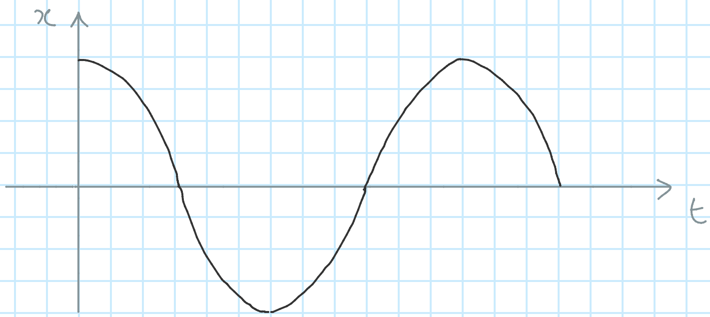
if $t = 0 \longrightarrow x(0) = B_1 \longrightarrow B_1 \equiv x_0$ initial position

Similarly

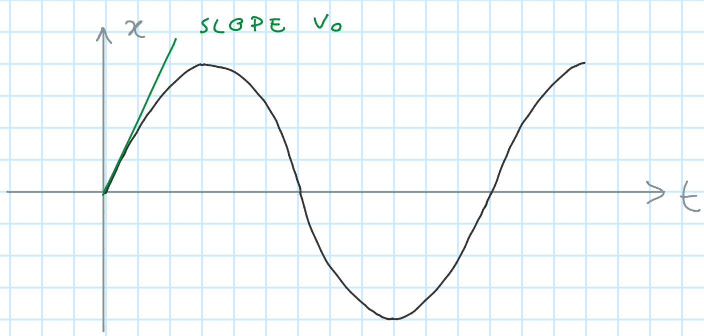
$$\frac{dx}{dt} = -B_1 \omega \sin(\omega t) + B_2 \omega \cos(\omega t)$$

$$\left.\frac{dx}{dt}\right|_{t=0} = B_2 \omega \equiv v_0$$ initial velocity

if $v_0 = 0$ $$\boxed{x(t) = x_0 \cos(\omega t)}$$



if $x_0 = 0$ $$\boxed{x(t) = \frac{v_0}{\omega} \sin(\omega t)}$$



The period of the functions above is

$$\omega T = 2\pi \qquad T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{k}}$$

$\longrightarrow$

The general solution, with non zero initial position and velocity can be rewritten in terms of an amplitude constant and a phase

$$A \equiv \sqrt{B_1^2 + B_2^2}$$

$$x(t) = A\left[\underbrace{\frac{B_1}{A}}_{\cos\delta}\cos(\omega t) + \underbrace{\frac{B_2}{A}}_{\sin\delta}\sin(\omega t)\right]$$

$$= A\left[\cos\delta\cos(\omega t) + \sin\delta\sin(\omega t)\right]$$

$$\boxed{x(t) = A\cos(\omega t - \delta)}$$

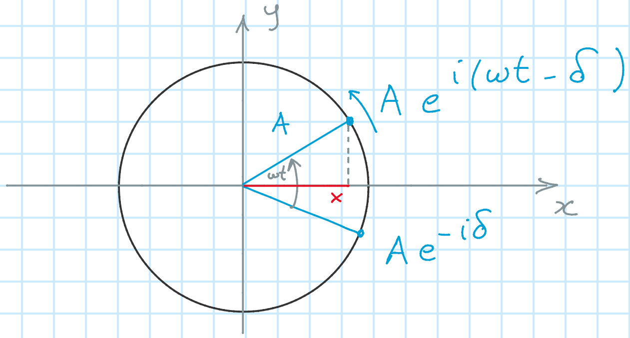
PHASE SHIFTED COSINE SOLUTION

Consequently, one can write x also as

$$\boxed{x(t) = \mathrm{Re}\left\{A\, e^{i(\omega t - \delta)}\right\}}$$

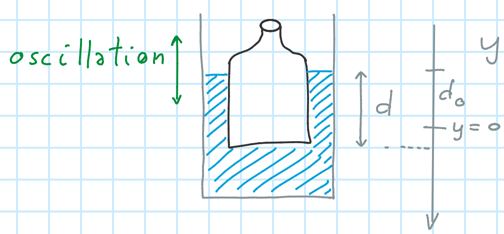
REAL PART OF A COMPLEX EXPONENTIAL

The complex number in square brackets rotates on a circle of radius A. The projection of this point on the x axis represents the position of the oscillating object.



Example (from Taylor, chapter 5, example 5.2)

A bottle is floating in a container of water. The bottle is pushed down to that the bottom of the bottle reaches a depth d in the water. The bottle is then released. What is the period of the bottle's oscillation? Neglect any sort of friction.



Assume the bottle to be cylindrical and the cross sectional area to be equal to A. When the bottle is at equilibrium and it is not oscillating, it reaches a depth that can be determined by observing that the weight of the bottle must be balanced by the buoyant force acting on the bottle.

$$m g = \rho g \underbrace{A d_0}_{\text{volume displaced}}$$

density of water

Now let's assumed that when the bottle is pushed down and then released, the bottom of the bottle reaches a depth d equal to

$$d = d_0 + y$$

The equation of motion is therefore

$$m \ddot{y} = \underbrace{m g}_{= + \rho g A d_0} - \rho g A (d_0 + y) \longrightarrow m \ddot{y} = - \rho g A y$$

$$\ddot{y} = - \frac{\rho g A}{m} y = - \frac{g}{d_0} y$$

since $m = \rho A d_0$

Therefore

$$\omega^2 = \frac{g}{d_0}$$

And $$\boxed{T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{d_0}{g}}}$$

The total mechanical energy of the oscillator can be easily determined starting from

$$x(t) = A \cos(\omega t + \delta) \qquad \dot{x}(t) = - A \omega \sin(\omega t - \delta)$$

$$E = T + U = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2 = \frac{A^2}{2} k \frac{m}{k} \omega^2 \sin^2(\omega t - \delta) + \frac{A^2}{2} k \cos^2(\omega t - \delta)$$

$$= \frac{1}{2} k A^2$$