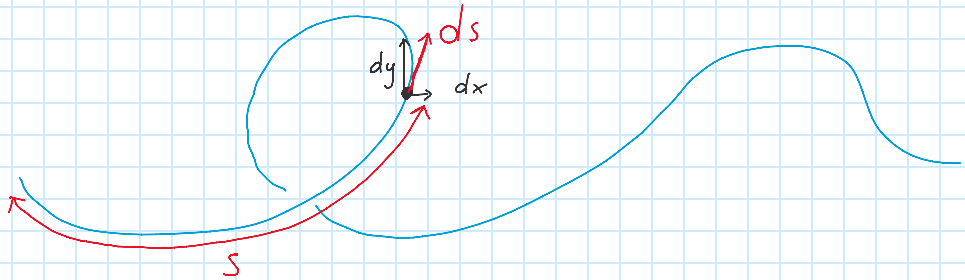
Curvilinear one dimensional systems

Tuesday, July 9, 2019 3:55 PM

Let's now consider a system that is one dimensional but not linear. A classic example is a bead that is free to slide on a wire. Let's call s the distance traveled by the bead on the wire.



$$\bar{v} = \frac{dx}{dt}\hat{\imath} + \frac{dy}{dt}\hat{\jmath} \qquad v^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{(dx)^2 + (dy)^2}{dt^2} = \frac{ds^2}{dt^2} = \left(\frac{ds}{dt}\right)^2$$

$$\boxed{v = \frac{ds}{dt}}$$

The speed of the bead is the time derivative of s, therefore the kinetic energy of the bead is

$$T = \frac{1}{2} m \left(\frac{ds}{dt}\right)^2 = \frac{1}{2} m \dot{s}^2$$

The normal force applied by the bead on the wire is what keeps the bead on the wire.

Let's now take the time derivative of the trivial identity

$$v^2 = \bar{v} \cdot \bar{v}$$

l.h.s $\quad \frac{d}{dt} v^2 = 2v \frac{dv}{dt} = 2v \frac{d^2 s}{dt^2}$

r.h.s $\quad \frac{d}{dt} \bar{v}\cdot\bar{v} = 2\bar{v}\cdot\frac{d\bar{v}}{dt} = 2\bar{v}\cdot\bar{a} = \frac{2}{m}\bar{v}\cdot\bar{F}$

$$= \frac{2}{m} v F \cos\vartheta = \frac{2}{m} v F_{tan}$$

tangential component of the force

angle between $\bar{v}$ and $\bar{F}$

$$2v\frac{d^2s}{dt^2} = \frac{2}{m} v F_{tan} \longrightarrow \boxed{m\ddot{s} = F_{tan}}$$

The normal force does not do any work. If all of the other forces acting on the bead are conservative one can write that

$$\bar{F} \cdot d\bar{s} = F_{tan}\, ds$$

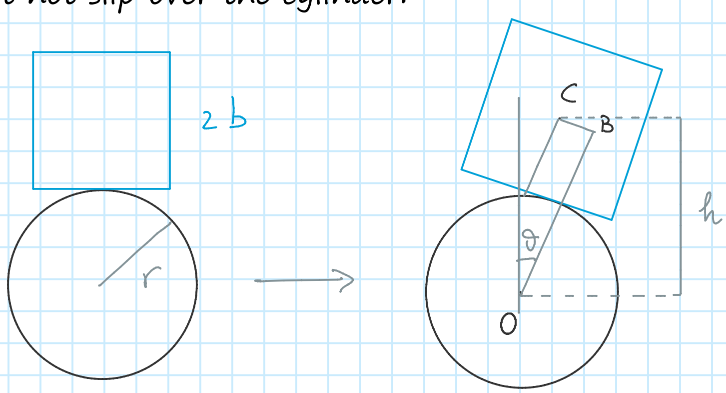
$$\bar{F} \cdot d\bar{s} = -\nabla U \cdot d\bar{s} = -dU$$

$$F_{tan}\, ds = -dU \qquad \boxed{F_{tan} = -\frac{dU}{ds}}$$

The mechanical energy is constant, minima in U(s) correspond to points of stable equilibrium, maxima in U(s) correspond to points of unstable equilibrium.

Example of a curvilinear one-dimensional system

Consider a cube placed on a fixed cylinder, where the cube can only rock back and forth but not slip over the cylinder.



When the cube is with its center on top of the center of the cylinder, is it in stable equilibrium?

The only force doing work here is gravity, one can write the potential energy as a function of the angle theta (we assume a small theta)

$$U(\theta) = mgh = mg\left(OB\cos\theta + BC\sin\theta\right)$$

$$= mg\left[(r+b)\cos\theta + \underbrace{r\theta}_{=BC \text{ for small } \theta}\sin\theta\right]$$

$$\frac{dU}{d\theta} = mg\left[-(r+b)\sin\theta + r\sin\theta + r\theta\cos\theta\right]$$

$$= mg\left[-b\sin\theta + r\theta\cos\theta\right]$$

$$\left.\frac{dU}{d\theta}\right|_{\theta=0} = 0 \quad \longrightarrow \quad \theta = 0 \text{ is an equilibrium position}$$

is it stable or not?

$$\frac{d^2U}{d\theta^2} = mg\left[r\cos\theta - r\theta\sin\theta - b\cos\theta\right]$$

$$= mg\left[(r-b)\cos\theta - r\theta\sin\theta\right]$$

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta=0} = mg(r-b)$$

→ if $r > b$ stable eq

→ if $r < b$ unstable eq.