Linear one dimensional systems

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Conservative one dimensional systems where an object can only move along a straight line have the special property that one can directly extract the equation of motion of an object from conservation of energy alone. Let's call x the coordinate along which the object can move.

$$E = \text{const} = T + U(x) = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + U(x)$$

$$\frac{dx}{dt} = \pm \sqrt{\frac{2}{m}} \sqrt{E - U(x)}$$

cons. of energy cannot know if the object is moving toward larger or smaller x

$$dt = \pm \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}$$

$$\int_{t_i}^{t_f} dt = \pm \int_{x_i}^{x_f} \sqrt{\frac{m}{2}} \frac{dx}{\sqrt{E - U(x)}}$$

Now let's take the initial t equal to 0, and let's rename the final time t

$$t = \pm \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{dx}{\sqrt{E - U(x)}}$$

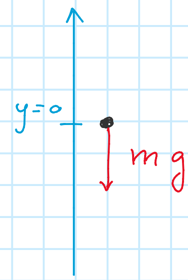
$+$ for $x > x_0$

$-$ for $x < x_0$

If one is now able to calculate this integral for the specific potential that one is looking at, one can find t(x). By inverting that result one can then find x(t).

Let's consider the simple example of an object in free fall. In addition, according to tradition, let's rename the coordinate y rather than x. Take the y axis going up and the initial position of the object to be y=0. When the object is dropped the total mechanical energy is zero and one needs to integrate from y=0 to a negative value of y.

$U = mgy \qquad E = 0$



$$t = -\sqrt{\frac{m}{2}} \int_0^{y} \frac{dy'}{\sqrt{-mgy'}}$$

$y' = -u$

$$t = -\frac{1}{\sqrt{2g}} \int_0^{-y} \frac{-du}{\sqrt{u}} = \frac{1}{\sqrt{2g}} \int_0^{-y} \frac{du}{\sqrt{u}}$$

$$t = \frac{2}{\sqrt{2g}} (-y)^{\frac{1}{2}} = \sqrt{\frac{2}{g}} \sqrt{-y}$$

$$t^2 = -\frac{2}{g} y \qquad \boxed{y = -\frac{1}{2} g t^2}$$

AS EXPECTED