Curl of a conservative force

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We defined a conservative force as a force that does the same amount of work on an object that moves from a given initial point to a given final point, irrespectively from the path followed by the object from the initial point to the final point.

$$\int_1^2 \bar{F} \cdot d\bar{r} = U(1) - U(2)$$

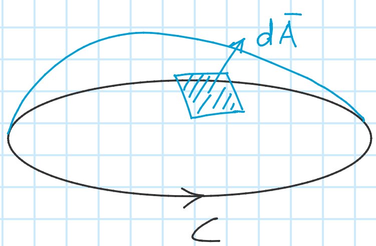
does not depend on the path but only on the location of the points 1 and 2

Consequently, the work done by a conservative force along a closed path is zero.

$$\oint_C \bar{F} \cdot d\bar{r} = 0$$

One can then apply a famous theorem of vector calculus, the Stokes' theorem

$$\oint_C \bar{F} \cdot d\bar{r} = \int_S \left(\nabla \times \bar{F}\right) \cdot d\bar{A}$$



rem

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$$

In the theorem above, S is any open surface bound by the closed curve C. Since the theorem applies to any closed curve C, one needs to conclude that the curl of the force is zero everywhere.

Consequently, a force is conservative if

$$\nabla \times \bar{F} = 0$$

This makes sense, since we already saw that for a conservative force

$$\bar{F} = -\nabla U \xrightarrow{\quad} \nabla \times \bar{F} = -\nabla \times (\nabla U) = 0$$

since

$$\nabla \times \nabla U = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x U & \partial_y U & \partial_z U \end{vmatrix} =$$

$$= \hat{i} \left(\cancel{\partial_y \partial_z U} - \cancel{\partial_z \partial_y U} \right) - \hat{j} \left(\cancel{\partial_x \partial_z U} - \cancel{\partial_z \partial_x U} \right)$$

$$+ \hat{k} \left(\cancel{\partial_x \partial_y U} - \cancel{\partial_y \partial_x U} \right) = 0$$

The curl of a gradient is always zero.

Let's prove that the curl of Coulomb force is zero (we will do the calculation using cartesian coordinates, one could prove this in a quicker way by using spherical coordinates, but let's avoid this potentially unfamiliar set of coordinates for the time being)

Coulomb force is

$$\bar{F} = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2} \hat{r} \equiv \gamma \frac{\bar{r}}{r^3}$$

$$\gamma \equiv \frac{q_1 q_2}{4\pi\varepsilon_0}$$

The x component of the curl of Coulomb's force will be

$$(\nabla \times \bar{F})_x = \partial_y F_z - \partial_z F_y = \frac{\partial}{\partial y}\left(\frac{\gamma z}{r^3}\right) - \frac{\partial}{\partial z}\left(\frac{\gamma y}{r^3}\right)$$

$$= \gamma z \frac{\partial}{\partial y}\left(\frac{1}{r^3}\right) - \gamma y \frac{\partial}{\partial z}\left(\frac{1}{r^3}\right) = (*)$$

rem $\quad r = \sqrt{x^2+y^2+z^2}$

$$(*) = \gamma z \left(-\frac{3}{\cancel{2}}\right) \frac{\cancel{2}y}{(x^2+y^2+z^2)^{\frac{5}{2}}} - \gamma y \left(-\frac{3}{\cancel{2}}\right) \frac{\cancel{2}z}{(x^2+y^2+z^2)^{\frac{5}{2}}}$$

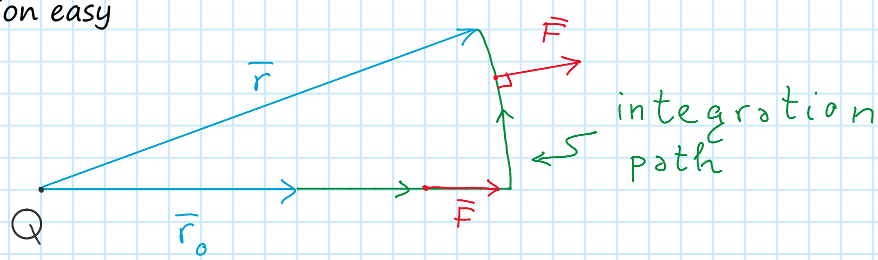
$$= -\frac{3\gamma}{r^5}(zy - yz) = 0$$

A similar calculation shows that all of the components of the curl of F are zero. Therefore Coulomb force is conservative.

What is the potential associated to Coulomb force? We need to integrate

$$U(\bar{r}) = -\int_{\bar{r}_0}^{\bar{r}} \bar{F} \cdot d\bar{r}$$

Since the integral above is path independent, let's choose a path that makes the calculation easy



Only the radial part of the path contributes to the integral

$$U(\bar{r}) = -\int_{r_0}^{r} \frac{\gamma}{(r')^2} \, dr' = \frac{\gamma}{r} - \frac{\gamma}{r_0}$$

Usually, the reference point r_0 is chosen at infinity, so that the electric potential energy becomes

$$\boxed{U(\bar{r}) = \frac{q_1 q_2}{4\pi\varepsilon_0} \frac{1}{r}}$$

Remember that when working in electrostatics one usually defines the potential (measured in volts) as the electric potential energy per unit of charge, so that for a point charge

$$\varphi(\bar{r}) = \frac{q_1}{4\pi\varepsilon_0} \frac{1}{r}$$

Notice that in that case the gradient of the potential gives the electric field in a point rather than a force

$$\bar{E}(\bar{r}) \equiv -\nabla\varphi(\bar{r})$$

Observe that in stating what are the conditions that need to be satisfied by a conservative force, we required that F depends only on the position and not on the velocity or time. What happens is one has a force with zero curl but with an explicit time dependence?

$$\bar{F} \equiv \bar{F}(\bar{r}, t) \qquad \text{with} \qquad \nabla\times\bar{F}(\bar{r},t) = 0 \ ?$$

One can still define a time dependent potential, but the mechanical energy is not conserved

$$\bar{F}(\bar{r},t) = -\nabla U(\bar{r},t)$$

but

$$\frac{dE}{dt} = \frac{d}{dt}(T+U) \neq 0$$

This can be easily proved, in fact:

$$dT = \frac{dT}{dt}dt = (m\dot{\bar{v}}\cdot\bar{v})\,dt = \bar{F}\cdot d\bar{r} \qquad (I)$$

$$dU = \underbrace{\frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy + \frac{\partial U}{\partial z}dz}_{\nabla U\cdot d\bar{r} = -\bar{F}\cdot d\bar{r}} + \frac{\partial U}{\partial t}dt$$

$$\bar{F}\cdot d\bar{r} = -dU + \frac{\partial U}{\partial t}dt$$

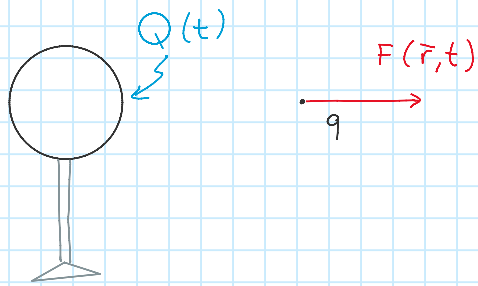
Now plug the above into (I)

$$dT = \bar{F}\cdot d\bar{r} = -dU + \frac{\partial U}{\partial t}\,dt$$

$$d(\underbrace{T+U}_{E}) = +\frac{\partial U}{\partial t}\,dt$$

$$\frac{dE}{dt} \neq 0$$

A simple situation of this kind is the case of an insulated conducting sphere leaking charge (for example, because of humidity) placed near a test charge q



If q is held still as Q decreases in time, the electric potential energy of the system decreases while the kinetic energy remains zero, therefore the mechanical energy of the system decreases in time. Of course total energy is still conserved; the mechanical energy lost goes into thermal energy since the leaking charge heats up the air surrounding the sphere.