Potential energy

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In general a force can depend on

1) The position of the object, e.g. Gravitational attraction between two objects
2) The velocity of the object, e.g. Air drag
3) The time at which the force is acting, e.g. Force applied by a time changing electric field

Conservative forces

For some forces a special property holds, namely the work done by the force on an object depends exclusively on the initial and final position of the object, not on the path followed by the object in going from the initial to the final position.

$$W(1 \to 2) = \int_1^2 \bar{F} \cdot d\bar{r}$$

the integral will give the same result no matter what is the path chosen

A force can be conservative only if it depends on the position of the object but not on Its velocity or on time.

Examples of conservative forces

1. Gravity $\quad \bar{F}(\bar{r}) = -G \frac{Mm}{r^2} \hat{r}$

2. Electrostatic force $\quad \bar{F}(\bar{r}) = q \bar{E}(\bar{r})$

Example of non conservative force: Friction

If an object is dragged on a flat floor, the friction force is constant in magnitude

$$\bar{F} = -\mu_K n \hat{v}$$

direction of $\bar{v}$

magnitude of the normal force

Consequently

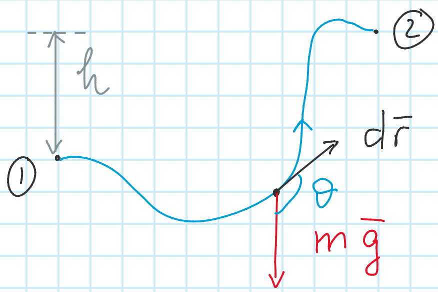
$$\int_1^2 \bar{F} \cdot d\bar{r} = -\mu_K n \int_1^2 \underbrace{\hat{v} \cdot d\bar{r}}_{\bar{v} \,//\, d\bar{r}} = -\mu_K n \int_1^2 d\ell$$

$$\int_1^2 \bar{F}\cdot d\bar{r} = -\mu_k n \ell$$

ℓ = length of the path

The work done by friction depends on the length of the path between points 1 and 2. Different paths correspond to different amounts of work.

On the contrary, gravity is a conservative force. Consider the case of an object moving near the surface of the earth, and calculate the work done by gravity on an object that moves from 1 to 2 along an arbitrary path



$$\int_1^2 \bar{F}\cdot d\bar{r} = \int_1^2 mg \underbrace{\cos\theta\, dr}_{-dy} = -\int_{y_1}^{y_2} mg\, dy =$$

$$= -mg\,(y_2 - y_1) = -mgh$$

The calculation does not depend on the shape of the path.

Potential energy

For conservative forces, it is possible to define a potential energy U as follows

$$U(\bar{r}) = -W(\bar{r}_0 \to \bar{r}) \equiv -\int_{\bar{r}_0}^{\bar{r}} \bar{F}(\bar{r}')\cdot d\bar{r}'$$

$\bar{r}_0$ = reference point

$\bar{r}$ = point of interest

Since the force F is assumed to be conservative, U depends only on the position of the object.

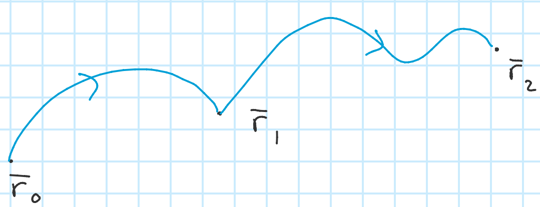
Mechanical energy

The mechanical energy of a point particle subject to conservative forces is defined as

$$E = T + U(\bar{r})$$

Conservation of mechanical energy

Let's now consider a point particle that is subject only to a conservative force and moves as shown in the figure



The work done by the conservative force will be

$$W(\bar{r}_0 \to \bar{r}_2) = W(\bar{r}_0 \to \bar{r}_1) + W(\bar{r}_1 \to \bar{r}_2)$$

$$W(\bar{r}_1 \to \bar{r}_2) = W(\bar{r}_0 \to \bar{r}_2) - W(\bar{r}_0 \to \bar{r}_1)$$

$$= -U(\bar{r}_2) + U(\bar{r}_1)$$

$$= -\left[U(\bar{r}_2) - U(\bar{r}_1)\right] = -\Delta U$$

One can now apply the work energy theorem to conclude that

$$\Delta T = T_2 - T_1 = W(\bar{r}_1 \to \bar{r}_2) = -\Delta U$$

$$\underbrace{T_2 + U_2}_{E_2} = \underbrace{T_1 + U_1}_{E_1}$$

The mechanical energy of the particle is the same at any two points in the trajectory. The mechanical energy is conserved, that is the reason why forces for which one can write a potential are called conservative forces.

Of course, the conservation of mechanical energy remains valid also when more than one conservative force acts on the object. (For example a mass attached to a spring will feel both the effect of gravity and Hooke's law)

Conservation of energy in presence of non conservative forces

When non conservative forces are acting on an object, mechanical energy is not conserved. However energy is still conserved. Non conservative forces transform mechanical energy in non-mechanical energy (for example heat and sound waves). One can still use the work energy theorem to write

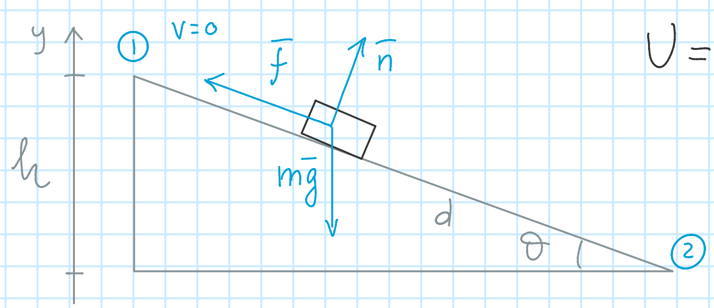
$$\Delta T = W = W_{cons} + W_{\substack{non \\ cons}} = -\Delta U + W_{\substack{non \\ cons.}}$$

$$\Delta (T + U) = W_{\substack{non \\ cons.}} \longrightarrow \boxed{\Delta E = W_{\substack{non \\ cons.}}}$$

The work done by non conservative forces equals the change in the mechanical energy of the object.

Block on an inclined plane

Let's solve a classic problem with the method of conservation of energy. Consider a block on an inclined plane in presence of friction. The block is sliding along the plane. The goal is to find the velocity with which the block will arrive at the bottom of the plane. The velocity must be written as a function of the length of the inclined plane, the angle that the plane makes with the horizontal direction, and the coefficient of kinetic friction.



$$U = mgy$$

$$\Delta E = E_2 - E_1 = \frac{1}{2} m v^2 - m g h = W_{friction}$$

$$\frac{1}{2} m v^2 - m g h = - f d = - \mu_k n d = - \mu_k m g d \cos\vartheta$$

$$\frac{v^2}{2} - g h = - \mu_k g d \cos\vartheta$$

$$v^2 = 2 g h - 2\mu_k g d \cos\vartheta = 2 g d \sin\vartheta - 2\mu_k g d \cos\vartheta$$

$$\boxed{v = \sqrt{2 g d (\sin\vartheta - \mu_k \cos\vartheta)}}$$