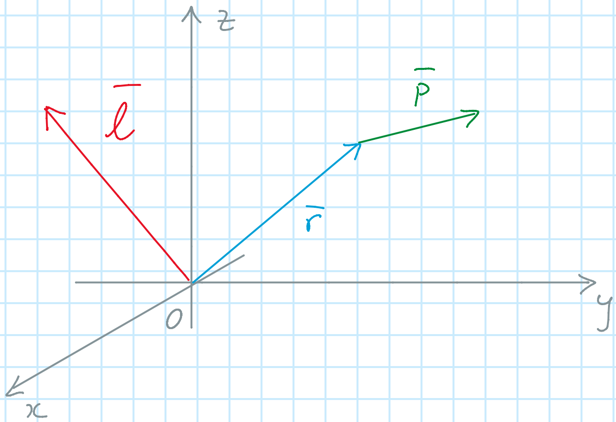
Angular momentum: Single particle

Wednesday, June 26, 2019 4:40 AM

The angular momentum of a single particle is defined as

$$\bar{\ell} = \bar{r} \times \bar{p}$$



Notice that the angular momentum depends on the choice of the origin of the frame of reference. Therefore, one should always specify what is the frame of reference with respect to which the angular momentum is calculated.

The time derivative of the angular momentum is

$$\dot{\bar{\ell}} = \frac{d}{dt}(\bar{r} \times \bar{p}) = \dot{\bar{r}} \times \bar{p} + \bar{r} \times \dot{\bar{p}}$$

but

$$\bar{p} = m\bar{v} = m\dot{\bar{r}} \longrightarrow \dot{\bar{r}} \times \bar{p} = m(\dot{\bar{r}} \times \dot{\bar{r}}) = 0 \; !$$

therefore

$$\dot{\bar{\ell}} = \bar{r} \times \dot{\bar{p}}$$

However, Newton's second law says that

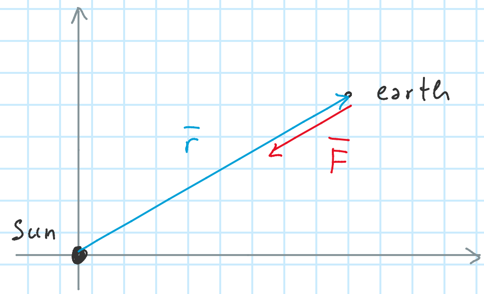
$$\dot{\bar{p}} = \bar{F} \longrightarrow \dot{\bar{\ell}} = \bar{r} \times \dot{\bar{p}} = \bar{r} \times \bar{F} = \bar{\tau}$$

TORQUE

$$\boxed{\dot{\bar{\ell}} = \bar{\tau}}$$

ROTATIONAL FORM OF NEWTON'S 2nd LAW

Also the torque depends on the choice of the origin of the frame of reference. It is often possible to choose the frame of reference in such a way that the total torque on a point particle is zero. In that case then the angular momentum of the particle is constant. For example consider the gravitational force applied by the sun on the earth, it makes sense to choose the origin of the frame of reference at the location of the sun.



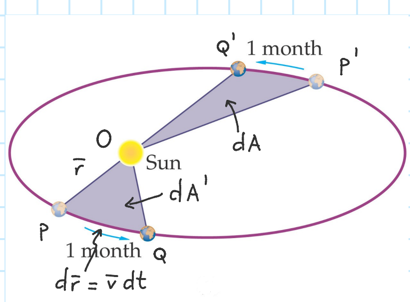
$$\bar{F} = -G\frac{M_S M_E}{r^2}\hat{r}$$

Since r and F are antiparallel the torque is zero and the angular momentum of the earth is constant. This is a general property of central forces. Since the angular momentum is constant, r and p always stay in the same plane. The orbit of a planet is confined to a two dimensional plane.

Kepler's second law

Kepler was able to describe planetary motion, by summarizing many years of observational data in three simple laws. Kepler's laws can be derived from Newton's laws of motion and from Newton's law for universal gravitation. Kepler's second law depends only on the fact that the gravitational force is a central force. The law says

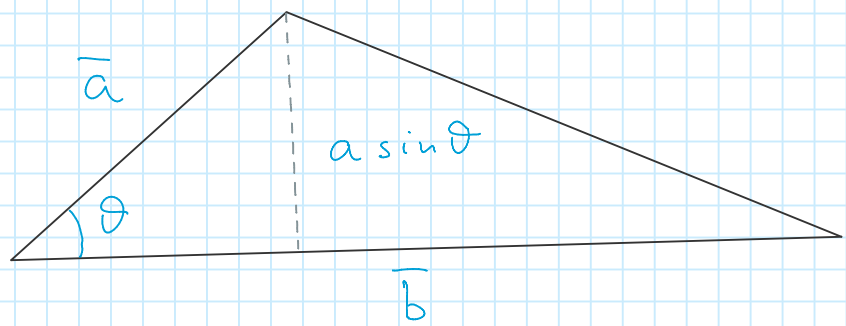
> As a planet moves around the sun, an imaginary line joining the planet to the sun sweeps out equal areas in equal times.



In order to prove the law let's start by observing that if two sides of a triangle are the vectors a and b, the area of the triangle is

$$A = \frac{1}{2} |\bar{a} \times \bar{b}|$$

Indeed



$$A = \frac{1}{2} b (a \sin\vartheta) = \frac{1}{2} |\bar{a} \times \bar{b}|$$

Consider now a small (infinitesimal) motion of a planet. The area of the "triangle" OPQ in the figure can be written as

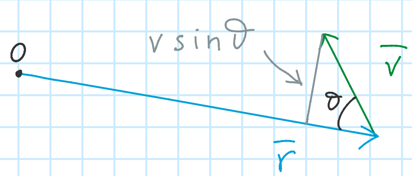
$$dA = \frac{1}{2} |\bar{r} \times \bar{v}\, dt| = \frac{1}{2m} |\bar{r} \times \bar{p}\, dt|$$

$$\frac{dA}{dt} = \frac{1}{2m} |\bar{r} \times \bar{p}| = \frac{\ell}{2m}$$

Since the planet angular momentum calculated with respect to the sun is constant, the derivative of the spanned area with respect to time is constant, which is precisely the statement of Kepler's second law.

An alternative way of proving the theorem proceeds as follows

$$\ell = |\bar{r} \times \bar{p}| = m |\bar{r} \times \bar{v}| = m r \underbrace{v \sin\vartheta}_{v_\perp}$$



$$\ell = m r v_\perp = m r \, r\omega = m r^2 \omega$$

In addition

$$dA = \frac{1}{2} r \, v \sin\vartheta \, dt = \frac{1}{2} r^2 \omega \, dt$$

$$\frac{dA}{dt} = \frac{1}{2} r^2 \omega = \frac{\ell}{2m}$$

Notice that the first equality above shows that if r decreases the angular velocity must grow and vice versa, since the derivative of the area with respect to time is constant.