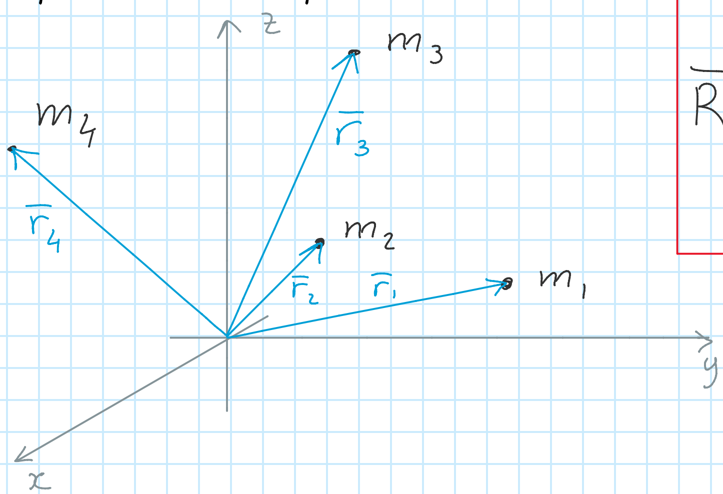
Center of mass

Wednesday, June 26, 2019 3:32 AM

When discussing the dynamics of a system of particles it is useful to introduce the concept of center of mass. Consider a system of N particles and a frame of reference used to describe the positions of those particles



$$\boxed{\bar{R} = \frac{\sum_{i=1}^{N} m_i \bar{r}_i}{\sum_{i=1}^{N} m_i} = \frac{1}{M}\sum_{i=1}^{N} m_i \bar{r}_i}$$

CENTER OF MASS

$$M \equiv \sum_{i=1}^{N} m_i$$

The equation that defines the position of the center of mass is a vector equation, so that it is possible to split into three equations, one for each coordinate of the center of mass:

$$X = \frac{1}{M}\sum_{i=1}^{N} m_i x_i \quad , \quad Y = \frac{1}{M}\sum_{i=1}^{N} m_i y_i \quad , \quad Z = \frac{1}{M}\sum_{i=1}^{N} m_i z_i$$

The center of mass is the weighted average of the particle positions, where the masses are the weights. Massive points "pull" the center of mass toward them. For example, the center of mass of the system sun-earth is close to the sun.

The total momentum of a system of particles is the momentum that a particle with the total mass of the system, moving at the speed of the center of mass would have.

$$\bar{P} = \sum_{i=1}^{N} \bar{p}_i = \sum_{i=1}^{N} m_i \dot{\bar{r}}_i = M \dot{\bar{R}}$$

An interesting consequence of this is that the center of mass of the system moves as if the total force acting on the system was applied to a particle of mass M located in the center of mass:

$$\dot{\bar{P}} = \bar{F}_{ext} \longrightarrow \dot{\bar{P}} = \boxed{M \ddot{\bar{R}} = \bar{F}_{ext}}$$

NEWTON'S 2nd LAW FOR THE C.O.M.

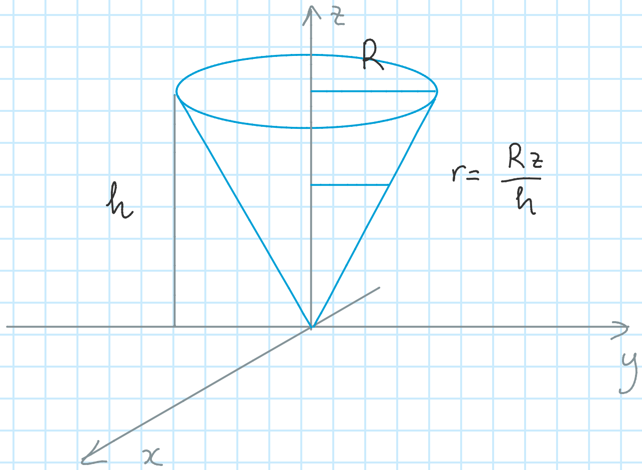
This is the reason why, when the dimensions of an object are small compared to the distances traveled by the object, one can efficiently model an extended object with a point particle located in Its center of mass.

Center of mass for extended objects

Objects are made of molecules, so that in principle, it is correct to calculate the position of an object center of mass as a discrete sum over the molecules, as done above. However, in most cases it is more convenient to consider an object as a continuous distribution of mass. In this case one should replace summations with integrals as follows

$$\bar{R} = \frac{1}{M}\int_V \bar{r}\, dm = \frac{1}{M}\int_V \rho(\bar{r})\, \bar{r}\, dV$$

In the equation above V is the object's volume and rho is the object's density.
In order to become familiar with this way of calculating the center of mass of an object, let's consider the case of a cone of constant density



The integrals that one needs to calculate are

$$X = \frac{\rho}{M}\int_V x \underbrace{r\,dr\,d\phi\,dz}_{dV} = \frac{\rho}{M}\int_V r^2\cos\phi\,dr\,d\phi\,dz$$

$$= \frac{\rho}{M}\int_0^h dz \int_0^{\frac{Rz}{h}} dr\, r^2 \underbrace{\int_0^{2\pi} \cos\phi\,d\phi}_{=0} = 0$$

$$Y = \frac{\rho}{M}\int_V y\, r\,dr\,d\phi\,dz = \frac{\rho}{M}\int_0^h dz \int_0^{\frac{Rz}{h}} dr\, r^2 \int_0^{2\pi} \sin\phi\,d\phi = 0$$

As it was possible to guess from symmetry, the center of mass is located on the z axis. The z coordinate of the center of mass is located at

$$Z = \frac{\rho}{M}\int_V z\, r\,dr\,d\phi\,dz = \frac{\rho}{M}\int_0^h z\,dz \int_0^{\frac{Rz}{h}} dr\, r \int_0^{2\pi} d\phi$$

$$= \frac{\rho}{M}\,2\pi \int_0^h dz\, z\,\frac{1}{2}\left(\frac{Rz}{h}\right)^2 = \frac{\pi\rho}{M}\,\frac{R^2}{h^2}\int_0^h dz\, z^3$$

$$= \frac{\pi\rho}{M}\,\frac{R^2}{\cancel{h^2}}\,\frac{h^{\cancel{4}\,2}}{4} = (*)$$

rem $\rho = \underbrace{\frac{M}{\frac{\pi}{3}R^2 h}}_{\text{cone's volume}}$

$$Z = (*) = \frac{\cancel{\pi}}{\cancel{M}}\,\frac{\cancel{M}}{\frac{\cancel{\pi}}{3}\cancel{R^2 h}}\,\frac{\cancel{R^2}}{4}\,h^{\cancel{2}} = \frac{3}{4}h$$

Therefore

$$\boxed{X = Y = 0 \qquad Z = \frac{3}{4}h}$$

CONE'S CENTER OF MASS

In case you want to check the formula for the cone's volume:

$$V = \int_V r\,dr\,d\phi\,dz = \int_0^h dz \int_0^{\frac{Rz}{h}} dr\, r \int_0^{2\pi} d\phi = 2\pi\int_0^h dz\,\frac{1}{2}\,\frac{R^2 z^2}{h^2}$$

$$= \pi\frac{R^2}{h^2}\int_0^h dz\, z^2 = \frac{\pi R^2}{3h^2}\,h^3 = \frac{\pi R^2 h}{3}$$

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