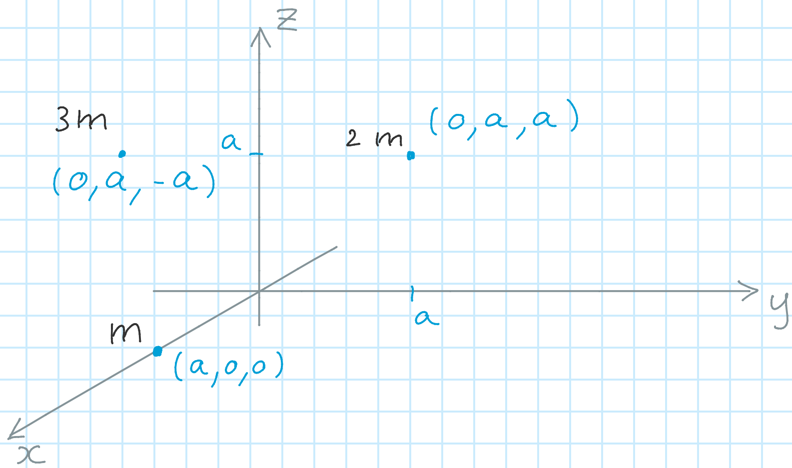
Principal axes - 2

Thursday, December 12, 2019 9:39 AM

Taylor, problem 10.35

A rigid body consists of three masses fastened as follows: m at (a,0,0), 2m at (0,a,a) and 3m at (0,a,-a). Find the inertia tensor I. Find the principal moments and a set of orthogonal principal axes.



$$I_{yz} = -\sum_{i=1}^{3} m_i y_i z_i = -2ma^2 + 3ma^2 = ma^2$$

$$I_{xy} = -\sum_{i=1}^{3} m_i x_i y_i = 0$$

$$I_{xz} = -\sum_{i=1}^{3} m_i x_i z_i = 0$$

$$I_{xx} = \sum_{i=1}^{3} m_i (y_i^2 + z_i^2) = 2m(a^2+a^2) + 3m(a^2+a^2) = 10ma^2$$

$$I_{yy} = \sum_{i=1}^{3} m_i (x_i^2 + z_i^2) = ma^2 + 2ma^2 + 3ma^2 = 6ma^2$$

$$I_{zz} = \sum_{i=1}^{3} m_i (x_i^2 + y_i^2) = ma^2 + 2ma^2 + 3ma^2 = 6ma^2$$

$$I = ma^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & 1 \\ 0 & 1 & 6 \end{pmatrix}$$

eigen values

$$\det(I - \lambda \mathbb{1}) = 0$$

$\lambda \propto ma^2$, set $ma^2 = 1$ in the intermediate stages

$$\det(I - \lambda \mathbb{1}) = \underbrace{(10 - \lambda)}_{\text{zero for } \lambda = 10}\left[(6-\lambda)^2 - 1\right] = 0$$

$$(6-\lambda)^2 = 1 \qquad 6 - \lambda = \pm 1 \quad \begin{matrix} \nearrow \lambda = 5 \\ \searrow \lambda = 7 \end{matrix}$$

$$\lambda_1 = 10 \; ma^2 \qquad \lambda_2 = 7 ma^2 \qquad \lambda_3 = 5 ma^2$$

eigenvectors

$$\lambda = \lambda_1 \qquad (I - \lambda_1 \mathbb{1}) \cdot \overline{\omega} = 0$$

$$ma^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & 1 \\ 0 & 1 & -4 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0 \qquad \begin{matrix} -4\omega_y + \omega_z = 0 \\ \omega_y - 4\omega_z = 0 \end{matrix}$$

$$\longrightarrow \quad \omega_y = \omega_z = 0 \qquad \boxed{\hat{e}_1 = \hat{\imath}}$$

$$\lambda = \lambda_2 \qquad (I - \lambda_2 \mathbb{1}) \cdot \overline{\omega} = 0$$

$$ma^2 \begin{pmatrix} 3 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\omega_x = 0 \qquad \omega_y = \omega_z \qquad \boxed{\hat{e}_2 = \frac{1}{\sqrt{2}}(\hat{\jmath} + \hat{k})}$$

$\lambda = \lambda_3 \qquad (I - \lambda_3 \mathbb{1}) \cdot \overline{\omega} = 0$

$$ma^2 \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$\omega_x = 0 \qquad \omega_y = -\omega_z \qquad \boxed{\hat{e}_3 = \frac{1}{\sqrt{2}} (\hat{j} - \hat{k})}$