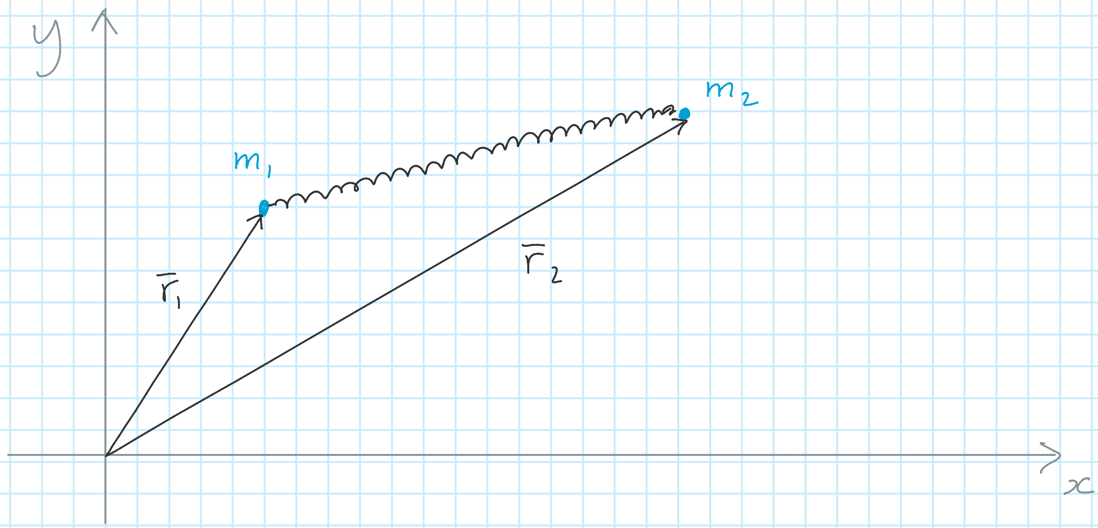
Hamiltonian two body problem

Tuesday, November 19, 2019 9:32 AM

Taylor, problem 13.21

Two masses m1 and m2 are joined by a massless spring (force constant k and natural length l_0) and are confined to move in a frictionless horizontal plane. Use the center of mass and relative positions to write the Hamiltonian. Figure out which coordinates are ignorable and which one are not ignorable. Write down the 8 Hamilton's equations of motion. Solve the equations for the special case that the generalized momentum associated to the polar angle of the relative position is zero.



$$\bar{R} \equiv \frac{m_1\bar{r}_1 + m_2\bar{r}_2}{\underbrace{m_1 + m_2}_{=M}}, \quad \bar{r} \equiv \bar{r}_1 - \bar{r}_2, \quad \bar{r}_1 = \bar{R} + \frac{m_2}{M}\bar{r}, \quad \bar{r}_2 = \bar{R} - \frac{m_1}{M}\bar{r}$$

$$T = \frac{1}{2}m_1\dot{\bar{r}}_1^{\,2} + \frac{1}{2}m_2\dot{\bar{r}}_2^{\,2} = \frac{1}{2}M\dot{\bar{R}}^2 + \frac{1}{2}\mu\dot{\bar{r}}^2 \qquad \mu \equiv \frac{m_1 m_2}{M}$$

$$= \frac{M}{2}\Big(\underbrace{\dot{X}^2 + \dot{Y}^2}_{\text{c.o.m. cartesian coordinates}}\Big) + \frac{\mu}{2}\Big(\underbrace{\dot{r}^2 + r^2\dot{\phi}^2}_{\text{relative position polar coordinates}}\Big)$$

$$P_x = \frac{\partial T}{\partial \dot{x}} = M\dot{x} \qquad P_y = \frac{\partial T}{\partial \dot{y}} = M\dot{y}$$

$$p_r = \frac{\partial T}{\partial \dot{r}} = \mu \dot{r} \qquad p_\phi = \frac{\partial T}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}$$

$$\mathcal{H} = \frac{P_x^2 + P_y^2}{2M} + \frac{p_r^2}{2\mu} + \frac{p_\phi^2}{2\mu r^2} + \frac{1}{2} k (r - l_0)^2$$

$$\boxed{X, Y, \phi \text{ are ignorable coordinates}}$$

$$\dot{X} = \frac{\partial H}{\partial P_x} = \frac{P_x}{M} \qquad \dot{P}_x = -\frac{\partial \mathcal{H}}{\partial X} = 0$$

$$\dot{Y} = \frac{\partial \mathcal{H}}{\partial P_y} = \frac{P_y}{M} \qquad \dot{P}_y = -\frac{\partial \mathcal{H}}{\partial Y} = 0$$

$$\dot{\phi} = \frac{\partial \mathcal{H}}{\partial p_\phi} = \frac{p_\phi}{\mu r^2} \qquad \dot{p}_\phi = -\frac{\partial \mathcal{H}}{\partial \phi} = 0$$

$$\dot{r} = \frac{\partial \mathcal{H}}{\partial p_r} = \frac{p_r}{\mu} \qquad \dot{p}_r = -\frac{\partial \mathcal{H}}{\partial r} = \frac{p_\phi^2}{\mu r^3} - k(r - l_0)$$

Effectively, this problem only has one degree of freedom, which is r. If the angular momentum p_ϕ is zero, then the equation of motion for r is just the equation of motion of a harmonic oscillator. If the angular momentum is not zero then there is also the contribution of the "centrifugal force".