1/r^3 correction

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Taylor problem 8.23

Solve the gravitational two body problem by adding a 1/r^3 correction to the force

$$F = -\frac{k}{r^2} + \frac{\lambda}{r^3}$$

On general ground we found that the equation of motion for the two body problem in presence of a central force is

$$\frac{d^2u}{d\phi^2} = -u - \frac{\mu}{\ell^2 u^2} F \qquad \text{with} \quad u = \frac{1}{r}$$

$$u'' = -u - \frac{\mu}{\ell^2 u^2}\left(-k u^2 + \lambda u^3\right)$$

$$u'' = -u + \frac{\mu}{\ell^2} k - \lambda \frac{\mu}{\ell^2} u$$

$$u'' = -\left(1 + \lambda \frac{\mu}{\ell^2}\right) u + \frac{\mu}{\ell^2} k$$

<u>set</u> $\quad w = \left(1 + \lambda \frac{\mu}{\ell^2}\right) u - \frac{\mu}{\ell^2} k$

$$w'' = \left(1 + \lambda \frac{\mu}{\ell^2}\right) u''$$

$$w'' = -\left(1 + \lambda \frac{\mu}{\ell^2}\right) w$$

set $\quad \beta^2 = 1 + \lambda \frac{\mu}{\ell^2}$

$$\left[\lambda \frac{\mu}{\ell^2}\right] = \overbrace{\cancel{kg}\,\frac{m}{s^2}\, m^{\cancel{3}}}^{[\lambda]} \; \frac{\cancel{kg}}{\cancel{kg^2}\,\frac{m^2}{s^2}\, m^2} = 1$$

$$w = A \cos\left(\beta\phi - \delta\right)$$

$$u = A\cos(\beta\phi - \delta) + \frac{\mu}{\ell^2}k$$

Choose $\delta = 0$ by orienting appropriately the x axis

$$u = A\cos(\beta\phi) + \underbrace{\frac{\mu}{\ell^2}k}_{\equiv \frac{1}{c}}$$

set $$A \equiv \frac{\epsilon}{c}$$

$$u = \frac{1}{c}\left(1 + \epsilon\cos(\beta\phi)\right)$$

$$\boxed{r(\phi) = \frac{c}{1 + \epsilon\cos(\beta\phi)}}$$

For $0 < \varepsilon < 1$ the orbit is bound, because the denominator of the equation above cannot go to zero. However, for arbitrary β the orbit is not closed.

$$r(\phi + 2\pi) \neq r(\phi)$$ for arbitrary β

however, if

$$m\beta 2\pi = n 2\pi \qquad m, n \in \mathbb{N}$$ $\rightarrow$ closed orbit if β is rational

if β goes to zero r does not depend on ϕ and the orbit is circular.