Two body problem with a spring

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Problem 8.9 in Taylor's book.

Consider two particles of equal mass m attached to each other by a light straight spring of force constant k and of natural length L and free to slide over a frictionless horizontal table.

a) Write down the Lagrangian in polar coordinates, in terms of the relative position and of the location of the COM.
b) Write down and solve Lagrange equations for the coordinates of the COM.
c) Write down Lagrange equations for the polar coordinates of the relative position. Solve them for the special cases in which either r or ϕ remain constant.

part a

$$\mathcal{L} = \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \mu \dot{\vec{r}}^2 - \frac{1}{2} k (r-L)^2$$

here $M = 2m$ $\mu = \frac{m}{2}$

$$\mathcal{L} = m \dot{\vec{R}}^2 + \frac{1}{4} m \dot{\vec{r}}^2 - \frac{1}{2} k (r-L)^2$$

part b

$$\frac{\partial \mathcal{L}}{\partial X} = 0 \qquad m \dot{X} \equiv P_x = \text{const}$$

$$\frac{\partial \mathcal{L}}{\partial Y} = 0 \qquad m \dot{Y} \equiv P_Y = \text{const}$$

part c

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{1}{2} m r \dot{\phi}^2 - k(r-L)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = \frac{1}{2} m \dot{r}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = \frac{1}{2} m \ddot{r} - \frac{1}{2} m r \dot{\phi}^2 + k(r-L) = 0$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{1}{2} m r^2 \dot{\phi} \equiv L = \text{const}$$

if $\phi = \text{const} \rightarrow \dot{\phi} = 0$, so that the eq for r becomes

$$\frac{1}{2} m \ddot{r} = -k(r-L)$$

$$\ddot{r} = -\frac{2k}{m}(r-L) \qquad \rightarrow \qquad \omega^2 = \frac{2k}{m} \checkmark$$

if $r = \text{const} \rightarrow \dot{r} = 0$

$$-\frac{1}{2} m r \dot{\phi}^2 + k(r-L) = 0$$

$$\dot{\phi}^2 = \frac{2k}{m} \frac{r-L}{r}$$