Two body angular momenta

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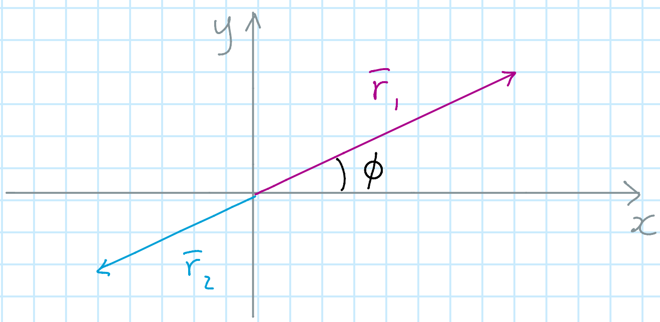
Problem 8.6 in Taylor's book.
Calculate the angular momenta of the two particles in the two body problem as a function of the total angular momentum of the system in the COM frame. Show that in the COM frame the two momenta are separately conserved.

$$\mathcal{L} = \underbrace{\frac{1}{2} M \dot{\bar{R}}^2}_{= 0 \text{ in c.o.m.}} + \frac{1}{2} \mu \dot{\bar{r}}^2 - U(r)$$

$$\mathcal{L} = \frac{1}{2} \mu (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r)$$

$$L = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \mu r^2 \dot{\phi}$$

For the individual particles one finds



$$m_1 \bar{r}_1 = - m_2 \bar{r}_2$$

$$\bar{\ell}_1 = \bar{r}_1 \times m_1 \dot{\bar{r}}_1$$

similarly

$$\ell_1 = m_1 r_1^2 \dot{\phi}^2$$
$$\ell_2 = m_2 r_2^2 \dot{\phi}^2$$

$\dot{\phi}$ is the same for both

$$m_1 r_1 = m_2 r_2 \qquad r = r_1 + r_2$$

rem $\bar{r} = \bar{r}_1 - \bar{r}_2$
but in magnitude
$r = r_1 + r_2$

$$r = r_1 + \frac{m_1}{m_2} r_1 = \frac{M}{m_2} r_1$$

$$r_1 = \frac{m_2}{M} r$$

$$\ell_1 = m_1 r_1^2 \dot{\phi} = m_1 \frac{m_2^2}{M^2} r^2 \dot{\phi} = \frac{m_2}{M} \mu r^2 \dot{\phi}$$

$$= \frac{m_2}{M} L \quad ✓$$

similarly

$$r_2 = \frac{m_1}{M} r$$

$$\ell_2 = m_2 r_2^2 \dot{\phi} = m_2 \frac{m_1^2}{M^2} r^2 \dot{\phi} = \frac{m_1}{M} \mu r^2 \dot{\phi} = \frac{m_1}{M} L \quad ✓$$

The conservation of total angular momentum implies that the angular velocity is constant. In the center of mass frame this is sufficient to guarantee that also the individual angular momenta of the two bodies are conserved.