Spring pendulum

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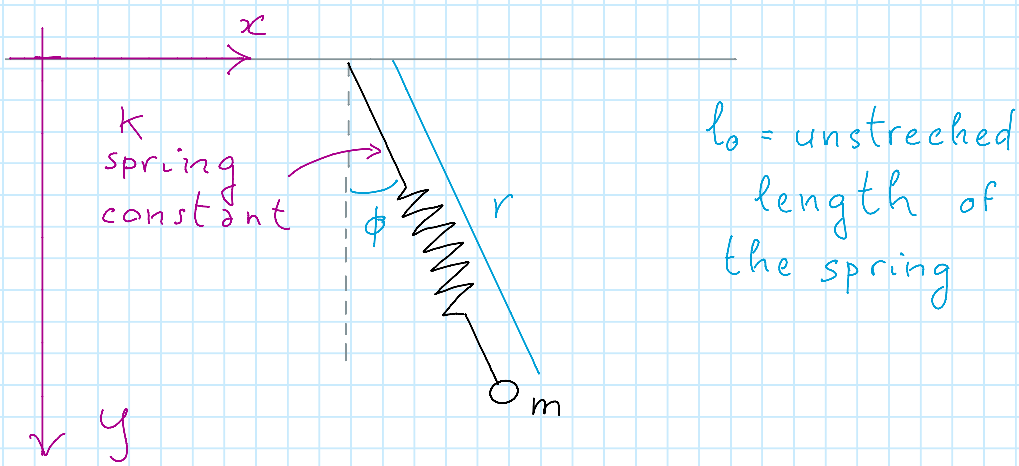
Problem 7.36 in Taylor

A pendulum is made of a massless spring that is suspended at one end from a fixed pivot O and has a mass m attached to the other end. The spring can stretch and compress but it cannot bend and the whole system is confined to a single vertical plane.

A) write down the Lagrangian of the pendulum, using as generalized coordinates the usual angle ϕ and the length r of the spring.

B) find the two Lagrange equations.

C) solve the equations in the case of small oscillations



$$\mathcal{L} = \frac{m}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - \frac{k}{2}\left(r - l_0\right)^2 + mgr\cos\phi$$

r and ϕ are functions of t

$$\frac{\partial \mathcal{L}}{\partial r} = m r \dot{\phi} - k(r - l_0) + mg\cos\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m\dot{r}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = m\ddot{r} - mr\dot{\phi}^2 + k(r - l_0) - mgr\cos\phi = 0$$

$$\boxed{m\ddot{r} - \underbrace{mr\dot{\phi}^2}_{\text{centripetal acceleration}} = \underbrace{-k(r-l_0)}_{\text{spring force}} - \underbrace{mg\cos\phi}_{\text{radial component of the weight}}}$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mgr\sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mr^2\dot{\phi}$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = 2mr\dot{r}\dot{\phi} + mr^2\ddot{\phi} + mgr\sin\phi = 0$$

$$\boxed{\ddot{\phi} = \underbrace{-\frac{g}{r}\sin\phi}_{\text{tangential component of the weight}} - 2\frac{\dot{r}}{r}\dot{\phi}}$$

Small oscillations

set $r = l + \epsilon$ where l is the equilibrium length of the spring when the mass m is hanging

$$mg = k(l - l_0)$$

the equation for r can then be rewritten in terms of ε as follows

$$m\ddot{\epsilon} + m(l+\epsilon)\dot{\phi}^2 = -k(l-l_0) - k\epsilon + mg\cos\phi$$

Now assume that ε and ϕ are small quantities of the same order and neglect quadratic terms

$$m\ddot{\epsilon} = -k(\ell-\ell_0) - k\epsilon + mg = -k\epsilon$$

$$\boxed{\ddot{\epsilon} = -\frac{k}{m}\epsilon}$$

harmonic
oscillator
equation

$$\boxed{\ddot{\phi} = -\frac{g}{r}\phi}$$

harmonic
oscillator
equation