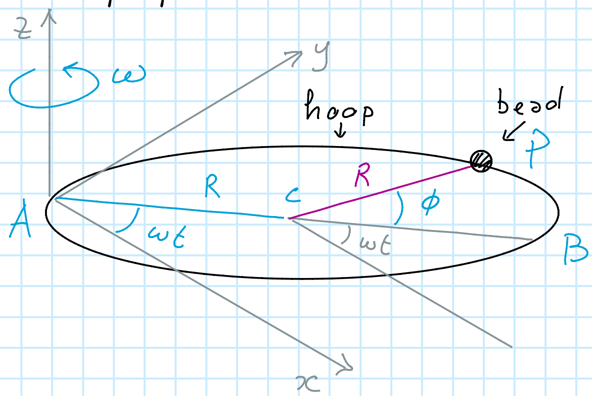
Bead on a horizontal spinning circle

Friday, October 30, 2020 4:19 PM

Problem 7.35 in Taylor's book.

A smooth horizontal wire hoop that is forced to rotate at a fixed angular velocity ω about a vertical axis through the point A. A bead of mass m is threaded on the hoop and is free to move around it, with its position specified by the angle ϕ that it makes at the center of the diameter AB. Find the Lagrangian for this system and show that the bead oscillates around the point B exactly like a simple pendulum.



ϕ is a function of time

$$x = R\cos(\omega t) + R\cos(\omega t + \phi)$$

$$y = R\sin(\omega t) + R\sin(\omega t + \phi)$$

$$\dot{x} = -R\omega\sin(\omega t) - R(\omega + \dot{\phi})\sin(\omega t + \phi)$$

$$\dot{y} = R\omega\cos(\omega t) + R(\omega + \dot{\phi})\cos(\omega t + \phi)$$

$$\mathcal{L} = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2)$$

$$= \frac{mR^2}{2}\Big[\omega^2 \sin^2(\omega t) + (\omega + \dot{\phi})^2 \sin^2(\omega t + \phi)$$

$$+ 2\omega(\omega + \dot{\phi})\sin(\omega t)\sin(\omega t + \phi)$$

$$+ \omega^2\cos^2(\omega t) + (\omega + \dot{\phi})^2\cos^2(\omega t + \phi)$$

$$+ 2\omega(\omega + \dot{\phi})\cos(\omega t)\cos(\omega t + \phi)\Big]$$

$$= \frac{mR^2}{2}\Big[\omega^2 + (\omega+\dot{\phi})^2 + 2\omega(\omega+\dot{\phi}) \times \big(\cos(\omega t)\cos(\omega t+\phi)$$

$$+ \sin(\omega t)\sin(\omega t+\phi)\big)\Big]$$

$$= \frac{mR^2}{2}\left[2\omega^2 + 2\omega\dot{\phi} + \dot{\phi}^2 + 2\omega(\omega+\dot{\phi})\cos\phi\right]$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = -mR^2\omega(\omega+\dot{\phi})\sin\phi$$

$$\frac{\partial \mathcal{L}}{\partial \dot{\phi}} = mR^2(\omega+\dot{\phi}) + mR^2\omega\cos\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = mR^2\left[\ddot{\phi} - \omega\dot{\phi}\sin\phi + \omega(\omega+\dot{\phi})\sin\phi\right] = 0$$

$$\ddot{\phi} - \cancel{\omega\dot{\phi}\sin\phi} + \omega^2\sin\phi + \cancel{\omega\dot{\phi}\sin\phi} = 0$$

$$\boxed{\ddot{\phi} = -\omega^2\sin\phi}$$

same equation as for the simple pendulum

the frequency of the oscillation around $\phi = 0$ is ω