Gradient in polar coordinates

Monday, October 19, 2020 9:04 AM

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$r = \sqrt{x^2+y^2} \qquad \phi = \arctan\frac{y}{x}$$

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}$$

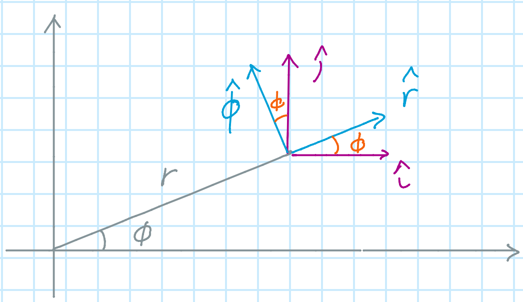
$$\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}$$

$$\frac{\partial r}{\partial x} = \frac{1}{2}\frac{2x}{\sqrt{x^2+y^2}} = \frac{r\cos\phi}{r} = \cos\phi$$

$$\frac{\partial r}{\partial y} = \frac{1}{2}\frac{2y}{\sqrt{x^2+y^2}} = \frac{r\sin\phi}{r} = \sin\phi$$

$$\frac{\partial \phi}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\phi}{r^2} = -\frac{\sin\phi}{r}$$

$$\frac{\partial \phi}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\frac{1}{x} = \frac{x}{x^2+y^2} = +\frac{r\cos\phi}{r^2} = \frac{\cos\phi}{r}$$



$$\hat{i} = \cos\phi\,\hat{r} - \sin\phi\,\hat{\phi}$$

$$\hat{j} = \sin\phi\,\hat{r} + \cos\phi\,\hat{\phi}$$

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$$

$s \equiv \sin\phi$

$c \equiv \cos\phi$

$$= (c\hat{r} - s\hat{\phi})\left(\frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi}\right)$$

$$+ (s\hat{r} + c\hat{\phi})\left(\frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi}\right)$$

$$= (c\hat{r} - s\hat{\phi})\left(c\frac{\partial}{\partial r} - \frac{s}{r}\frac{\partial}{\partial \phi}\right)$$

$$+ (s\hat{r} + c\hat{\phi})\left(s\frac{\partial}{\partial r} + \frac{c}{r}\frac{\partial}{\partial \phi}\right)$$

$$= \hat{r}\left(c^2\frac{\partial}{\partial r} - \frac{sc}{r}\frac{\partial}{\partial \phi} + s^2\frac{\partial}{\partial r} + \frac{cs}{r}\frac{\partial}{\partial \phi}\right)$$

$$+ \hat{\phi}\left(-sc\frac{\partial}{\partial r} + \frac{s^2}{r}\frac{\partial}{\partial \phi} + cs\frac{\partial}{\partial r} + \frac{c^2}{r}\frac{\partial}{\partial \phi}\right)$$

$$= \hat{r}\frac{\partial}{\partial r} + \hat{\phi}\frac{1}{r}\frac{\partial}{\partial \phi}$$