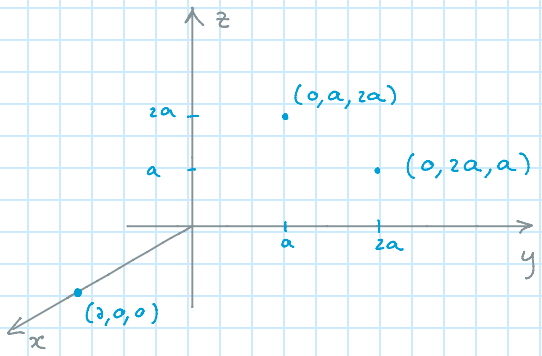
Principal axes - 1

Sunday, December 8, 2019 5:05 PM

Taylor, problem 10.36

Find the inertia tensor for the body shown in the figure. All three massive points have the same mass. Find the principal moments and a set of orthogonal principal axes.



$$I_{xx} = \sum_{i=1}^{3} m_i (y_i^2 + z_i^2) = 2m\left(4a^2 + a^2\right)$$

$$= 10\, m a^2$$

$$I_{yy} = \sum_{i=1}^{3} m_i (x_i^2 + z_i^2) = m^2 (a^2 + a^2 + 4a^2)$$

$$= 6\, m a^2$$

$$I_{zz} = I_{yy} = 6\, m a^2$$

$$I_{xy} = -\sum_{i=1}^{3} m_i x_i y_i = 0$$

$$I_{xz} = -\sum_{i=1}^{3} m_i x_i z_i = 0$$

$$I_{yz} = -\sum_{i=1}^{3} m_i y_i z_i = -m(2a^2 + 2a^2) = -4ma^2$$

$$I = m a^2 \begin{pmatrix} 10 & 0 & 0 \\ 0 & 6 & -4 \\ 0 & -4 & 6 \end{pmatrix}$$

eigenvalues / principal moments

$$\det(\mathbb{I} - \lambda \mathbb{1}) = 0$$

$$(10-\lambda)\left((6-\lambda)^2 + 16\right) = 0 \qquad \lambda_1 = 10\, ma^2$$

$$6 - \lambda = \pm 4 \quad \begin{cases} \nearrow \ \lambda_2 = 2\, ma^2 \\ \searrow \ \lambda_3 = 10\, ma^2 \end{cases}$$

eigenvectors

$$(\mathbb{I} - \lambda \mathbb{1})\,\bar{\omega} = 0$$

λ_1) λ_3)

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & -4 & -4 \\ 0 & -4 & -4 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\omega_y = -\omega_z$$

$$\bar{\omega}_1 = \frac{1}{\sqrt{2}}(\hat{j} - \hat{k}) \qquad \bar{\omega}_3 = \hat{i}$$

$$\bar{\omega}_1 \cdot \bar{\omega}_3 = 0$$

λ_2)

$$\begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & -4 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix} = 0$$

$$\omega_x = 0 \qquad \omega_y = \omega_z$$

$$\bar{\omega}_2 = \frac{1}{\sqrt{2}}(\hat{j} + \hat{k})$$