Precession in an axially symmetric body.

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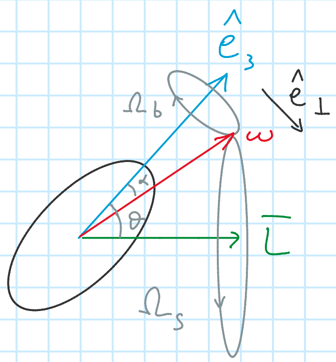
Taylor, problem 10.46

In the free precession of an axially symmetric body the vectors e_3, ω, and L stay on the same plane. Prove the relations

$$\Omega_s = \omega \frac{\sin\alpha}{\sin\vartheta} = \frac{L}{\lambda_1} = \omega \frac{\sqrt{\lambda_3^2 + (\lambda_1^2 - \lambda_3^2)\sin^2\alpha}}{\lambda_1}$$

Solution



$\bar{\Omega}_b = -\Omega_b \hat{e}_3$ angular vel of $\bar{\omega}$ w.r.t the body frame

$\bar{\Omega}_s = \Omega_s \frac{\bar{L}}{L}$ angular velocity of $\bar{\omega}$ w.r.t the space frame

$\bar{\omega}$ angular velocity of the body frame w.r.t the space frame

Angular velocities add up like regular velocities

$$\left[\bar{\Omega}_s\right]_{ws} = \left[\bar{\Omega}_b\right]_{wb} + \left[\bar{\omega}\right]_{bs}$$

multiply the relation above by a unit vector perpendicular to $\hat{e}_3$ (call it $\hat{e}_\perp$)

$$\hat{e}_\perp \cdot \bar{\Omega}_s = \cancel{\underbrace{\hat{e}_\perp \cdot \bar{\Omega}_b}_{\hat{e}_\perp \cdot \hat{e}_3 = 0}}^{\to 0} + \hat{e}_\perp \cdot \bar{\omega}$$

$$\hat{e}_\perp \cdot \bar{\omega} = \omega \cos\left(\frac{\pi}{2} - \alpha\right) = \omega \sin\alpha$$

$$\hat{e}_\perp \cdot \bar{L} = L\cos\left(\frac{\pi}{2}-\vartheta\right) = L\sin\vartheta$$

$$\hat{e}_\perp \cdot \bar{\Omega}_s = \Omega_s \sin\vartheta$$

$$\Omega_s \sin\vartheta = \omega \sin\alpha \qquad \boxed{\Omega_s = \omega\frac{\sin\alpha}{\sin\vartheta}}$$

$$\Omega_s = \frac{\omega\sin\alpha}{\sin\vartheta} = \frac{\omega_1}{\sin\vartheta} = \frac{\omega_1}{L_1/L} = \frac{L}{\lambda_1}$$

since $L_1 = \lambda_1\omega_1$

Finally

$$L^2 = \lambda_1^2\omega_1^2 + \lambda_2^2\omega_2^2 + \lambda_3^2\omega_3^2$$

but $\lambda_1 = \lambda_2$

$\omega_1 = \omega\sin\alpha\cos\varphi$

$\omega_2 = \omega\sin\alpha\sin\varphi$

$\omega_3 = \omega\cos\alpha$

$$L^2 = \left(\lambda_1^2\sin^2\alpha + \lambda_3^2\cos^2\alpha\right)\omega^2$$

$$= \left(\lambda_1^2\sin^2\alpha + \lambda_3^2 - \lambda_3^2\sin^2\alpha\right)\omega^2$$

$$\boxed{\Omega_s = \frac{L}{\lambda_1} = \omega\frac{\sqrt{\lambda_3^2 + (\lambda_1^2-\lambda_3^2)\sin^2\alpha}}{\lambda_1}}$$