Inertia tensor for a solid cone

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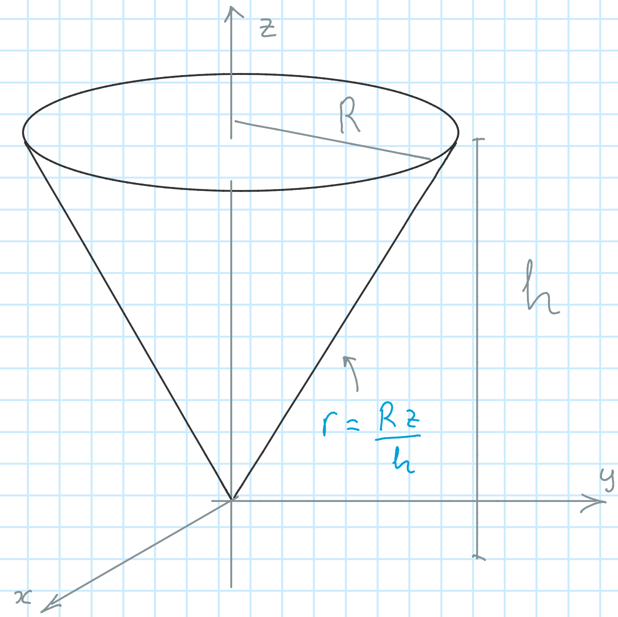
Here we want to consider the inertia tensor of a spinning top calculated by taking the origin at the tip of the spinning top and the z axis along the symmetry axis of the spinning top.



$$V = \int_0^h dz \int_0^{\frac{R}{h}z} dr\, r \int_0^{2\pi} d\varphi = 2\pi \int_0^h dz\, \frac{R^2 z^2}{2h^2} = \frac{\pi}{3} \frac{R^2}{h^2} h^3$$

$$= \frac{\pi}{3} R^2 h$$

$$\rho = \frac{M}{V} = \frac{3M}{\pi R^2 h}$$

$$I_{zz} = \rho \int_V dV \left(x^2 + y^2\right) = \rho \int_0^h dz \int_0^{\frac{R}{h}z} dr\, r \int_0^{2\pi} d\varphi \quad r^2$$

$$= 2\pi\rho \int_0^h dz\, \frac{1}{4} \frac{R^4 z^4}{h^4} = \frac{\pi}{10} \rho \frac{R^4}{h^4} h^5$$

$$= \frac{1}{10} \frac{3M}{R^2 h} R^4 h = \frac{3}{10} M R^2$$

Because of the symmetry with respect to rotations along the z axis, the components xx and yy of the inertia tensor are identical

$$I_{xx} = \int_V dV\, \rho\, (y^2 + z^2) = \int_V dV\, \rho\, y^2 + \int_V dV\, \rho\, z^2$$

$$\int_V dV\, \rho\, y^2 = \rho \int_0^h dz \int_0^{\frac{R}{h}z} dr\, r \underbrace{\int_0^{2\pi} d\varphi\, r^2 \sin^2\varphi}_{r^2\,\pi}$$

$$= \pi\rho \int_0^h dz\, \frac{1}{4}\, \frac{R^4}{h^4}\, z^4 = \frac{\pi \rho R^4}{20 h^4}\, h^5$$

$$= \frac{\pi}{20}\, \frac{3M}{\pi R^2 h}\, R^4 h = \frac{3}{20}\, M R^2$$

$$\int_V dV\, \rho\, z^2 = \rho \int_0^h dz\, z^2 \int_0^{\frac{R}{h}z} dr\, r \int_0^{2\pi} d\varphi$$

$$= \frac{2\pi}{2}\rho\, \frac{R^2}{h^2} \int_0^h dz\, z^4 = \pi\, \frac{3M}{\pi R^2 h}\, \frac{R^2}{h^2}\, \frac{h^5}{5}$$

$$= \frac{3}{5}\, M\, h^2$$

$$I_{xx} = I_{yy} = \frac{3}{20}\, M \left(R^2 + 4h^2\right)$$

All of the off diagonal elements of the inertia tensor are zero, since they all involve integral like

$$\int_0^{2\pi} d\varphi\, \cos\varphi = 0 \qquad \int_0^{2\pi} d\varphi\, \sin\varphi = 0$$

$$\int_0^{2\pi} d\varphi\, \sin\varphi \cos\varphi = 0$$

The reason for this is that the object has mirror symmetry with respect to the planes x=0 and y=0.

symm w.r.t $x=0 \longrightarrow I_{xy} = I_{xz} = 0$

symm w.r.t $y=0 \longrightarrow I_{yx} = I_{yz} = 0$

The inertia tensor for the cone, calculated with respect to the chosen frame of reference is

$$I = \frac{3}{20} M \begin{pmatrix} R^2+4h^2 & 0 & 0 \\ 0 & R^2+4h^2 & 0 \\ 0 & 0 & 2R^2 \end{pmatrix} \equiv \underbrace{\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}}_{\text{diagonal matrix}}$$

Consequently

$$\bar{L} = I\,\bar{\omega} = \lambda_1 \omega_x \hat{i} + \lambda_2 \omega_y \hat{j} + \lambda_3 \omega_z \hat{k}$$

Therefore, if the angular velocity points along one of the axes, the angular momentum is parallel to the angular velocity.