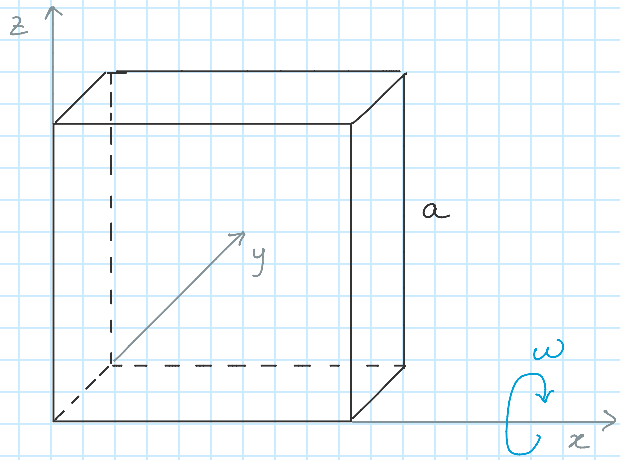
Inertia tensor for a solid cube

Sunday, November 24, 2019 2:58 PM

Rotation about an edge

Calculate the inertia tensor for a uniform solid cube of side a rotating about an edge of the cube. Identify the axis of rotation with the x axis, and calculate the angular momentum

$$\rho = \frac{M}{a^3}$$



$$I_{xx} = \underbrace{\int_0^a dx}_{a} \int_0^a dy \int_0^a dz \; \rho \; (y^2+z^2)$$

$$= a\rho \int_0^a dy \int_0^a dz \; (y^2+z^2) = 2a\rho a \frac{a^3}{3} = \frac{2}{3}\rho a^5 = \frac{2}{3} M a^2$$

By symmetry

$$I_{yy} = I_{zz} = I_{xx} = \frac{2}{3} M a^2$$

For the off diagonal elements one finds

$$I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz \; \rho \; xy$$

$$= -\rho \; \frac{a^2}{2} \frac{a^2}{2} a = -\frac{1}{4} M a^2$$

The tensor of inertia is therefore

$$I = Ma^2 \begin{pmatrix} \frac{2}{3} & -\frac{1}{4} & -\frac{1}{4} \\ -\frac{1}{4} & \frac{2}{3} & -\frac{1}{4} \\ -\frac{1}{4} & -\frac{1}{4} & \frac{2}{3} \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

Since the object is assumed to be rotating about the x axis, the angular momentum is

$$\bar{L} = \bar{I} \cdot \bar{\omega} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{Ma^2\omega}{12} \begin{pmatrix} 8 \\ -3 \\ -3 \end{pmatrix}$$

$$\bar{L} = \frac{Ma^2\omega}{12} \left(8\hat{i} - 3\hat{j} - 3\hat{k} \right)$$

Notice that L is not parallel to the vector angular velocity.

If the cube is instead rotating about the main diagonal, the angular momentum and the angular velocity will be pointing in the same direction. Indeed if the vector angular velocity is

$$\bar{\omega} = \frac{\omega}{\sqrt{3}} \left(\hat{i} + \hat{j} + \hat{z} \right)$$

$$\bar{L} = \bar{I} \cdot \bar{\omega} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$= \frac{Ma^2\omega}{12\sqrt{3}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \frac{Ma^2}{6} \bar{\omega}$$

Rotation about an axis going through the center of the cube

One can repeat the calculation of the inertia tensor for the case in which the axis of rotation goes through the center of the cube.

$$I_{xx} = \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \int_{-\frac{a}{2}}^{\frac{a}{2}} dz \; \rho \, (y^2+z^2) = 2\rho a^2 \left. \frac{y^3}{3} \right|_{-\frac{a}{2}}^{+\frac{a}{2}}$$

$$= \frac{2}{3}\rho a^2 \left(\frac{a^3}{8} - \left(-\frac{a^3}{8} \right) \right) = \frac{1}{6} \rho a^5 = \frac{1}{6} M a^2$$

$$I_{yy} = I_{zz} = I_{xx} = \frac{1}{6} M a^2$$

For the off diagonal elements

$$I_{xy} = - \int_{-\frac{a}{2}}^{\frac{a}{2}} dx \int_{-\frac{a}{2}}^{\frac{a}{2}} dy \; \rho \, x y = \rho \left[\frac{x^2}{2} \right]_{-\frac{a}{2}}^{+\frac{a}{2}} \left[\frac{y^2}{2} \right]_{-\frac{a}{2}}^{+\frac{a}{2}} = 0$$

$$I_{xy} = I_{yz} = I_{xz} = 0$$

All products of inertia are zero, because with the origin at the center of the cube and the axes parallel to the edges, the planes x=0, y=0, z=0 are axes of symmetry of the cube. Therefore

$$I = \frac{1}{6} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{6} M a^2 \underbrace{\mathbb{1}}_{3\times 3 \text{ unit matrix}}$$

In this case, the angular momentum is parallel to the angular velocity no matter what is the direction of the angular velocity

$$\bar{L} = \bar{I} \cdot \bar{\omega} = \frac{1}{6} M a^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \frac{1}{6} M a^2 \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix}$$

$$= \frac{1}{6} M a^2 \, \bar{\omega}$$

This is a consequence of the high symmetry of the cube around its center.