Canonical transformations

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Taylor problem 13.25

Under certain sets of transformations, which mix q and p and are called canonical transformations, and therefore are a different set of coordinates spanning the phase space, Hamilton's equations remain valid. For an Hamiltonian depending on a single generalized coordinate and momentum, an example of a canonical transformation is

$$q = \sqrt{2P}\,\sin Q \qquad\qquad p = \sqrt{2P}\,\cos Q$$

A) prove that

$$\text{if} \quad \dot{p} = -\frac{\partial \mathcal{H}}{\partial q} \quad \dot{q} = \frac{\partial \mathcal{H}}{\partial p} \longrightarrow \dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}, \quad \dot{Q} = \frac{\partial \mathcal{H}}{\partial P}$$

B) set k and m in the Hamiltonian of the harmonic oscillator equal to 1. Then prove that the canonical transformation listed above makes the coordinate Q ignorable.

Solution

A) $$\dot{q} = \sqrt{2P}\,\dot{Q}\cos Q + \cancel{\tfrac{1}{2}}\frac{\cancel{2}\dot{P}}{\sqrt{2P}}\sin Q$$

$$\dot{p} = -\sqrt{2P}\,\dot{Q}\sin Q + \cancel{\tfrac{1}{2}}\frac{\cancel{2}\dot{P}}{\sqrt{2P}}\cos Q$$

$$\frac{\partial \mathcal{H}}{\partial Q} = \underbrace{\frac{\partial \mathcal{H}}{\partial p}}_{\dot{q}}\frac{\partial p}{\partial Q} + \underbrace{\frac{\partial \mathcal{H}}{\partial q}}_{-\dot{p}}\frac{\partial q}{\partial Q} = -\dot{q}\sqrt{2P}\sin Q - \dot{p}\sqrt{2P}\cos Q$$

$$= -2P\dot{Q}\cancel{\cos Q\sin Q} - \dot{P}\sin^2 Q$$

$$+ 2P\dot{Q}\cancel{\sin Q\cos Q} - \dot{P}\cos^2 Q$$

$$= -\dot{P}$$

$$\frac{\partial \mathcal{H}}{\partial P} = \frac{\partial \mathcal{H}}{\partial p}\frac{\partial p}{\partial P} + \frac{\partial \mathcal{H}}{\partial q}\frac{\partial q}{\partial P} = \dot{q}\,\frac{\partial p}{\partial P} - \dot{p}\,\frac{\partial q}{\partial P}$$

$$= -\frac{1}{2}\sqrt{\frac{2}{P}}\sin Q\;\dot{p} \quad + \frac{1}{2}\sqrt{\frac{2}{P}}\cos Q \quad \dot{q}$$

$$= \dot{Q}\sin^2 Q - \frac{\dot{P}}{2P}\sin Q\cos Q$$

$$+ \dot{Q}\cos^2 Q + \frac{\dot{P}}{2P}\sin Q\cos Q$$

$$= \dot{Q}$$

$$\boxed{\dot{Q} = \frac{\partial \mathcal{H}}{\partial P}} \qquad \boxed{\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q}}$$

B) $$\mathcal{H}(q,p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{p^2}{2} + \frac{q^2}{2}$$

$$\mathcal{H}(Q,P) = \frac{2P\sin^2 Q}{2} + \frac{2P\cos^2 Q}{2} = P$$

P is the Hamiltonian, i.e. the energy

$$\dot{Q} = \frac{\partial \mathcal{H}}{\partial P} = 1 \longrightarrow Q(t) = t - t_0$$

$$\dot{P} = -\frac{\partial \mathcal{H}}{\partial Q} = 0 \longrightarrow P(t) = E$$

$$q(t) = \sqrt{2E}\,\sin(t - t_0)$$

$$p(t) = \sqrt{2E}\,\cos(t - t_0)$$