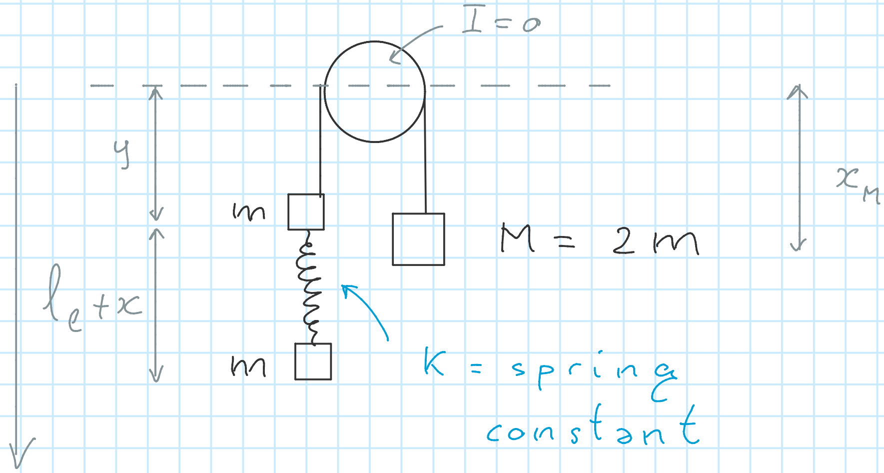
Atwood machine with spring

Tuesday, November 19, 2019 6:10 PM

Taylor problem 13.23

Consider the Atwood machine shown in the figure. Notice that x is the extension of the spring from its equilibrium length; that is the length of the spring is l_e + x where l_e is the equilibrium length (with all of the weights in position and M held stationary)., not to be confused with the natural length.



A) Show that the total potential energy (gravitational plus elastic) is just

$$U = \frac{1}{2} k x^2$$

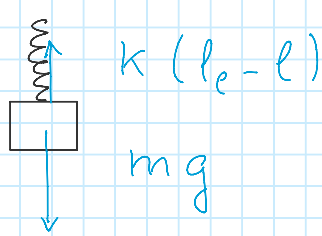
B) find the Hamiltonian and show that y is an ignorable coordinate.

C) solve the equations of motion for the initial conditions

$$x(0) = 0 \qquad y(0) = y_0 \qquad \dot{x}(0) = 0 \qquad \dot{y}(0) = 0$$

$\boxed{\text{Solution}}$

A) let's start by establishing the relation between the equilibrium length, k and the masses. At equilibrium



$$k(l_e - \underbrace{l}_{\text{natural length}}) = mg$$

Now let's calculate the total potential energy

$$x_M + y = \underset{\substack{\uparrow \\ \text{the rope has a fixed length}}}{c} \qquad x_M = c - y$$

$$U = -mgy - mg(y + l_e + x) - Mg x_M + \frac{1}{2}K(l_e + x - l)^2$$

$$= \cancel{-2mgy} - mgx + \cancel{\underbrace{M}_{=2m}gy} + \frac{1}{2}Kx^2 + K(l_e - l)x$$

$$+ \text{constants}$$

$$= \frac{1}{2}Kx^2 \cancel{- mgx} + \cancel{\underbrace{K(l_e - l)}_{mg}x} + \text{const.}$$

$$= \frac{1}{2}Kx^2 + \underbrace{\text{const.}}_{\text{set } = 0}$$

B) start by calculating the generalized momenta

$$T = \frac{1}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x} + \dot{y})^2 + \frac{1}{2}\underbrace{M\dot{x}_M^2}_{2m\dot{y}^2}$$

$$= \frac{3}{2}m\dot{y}^2 + \frac{1}{2}m(\dot{x} + \dot{y})^2$$

$$p_x = \frac{\partial T}{\partial \dot{x}} = m(\dot{x} + \dot{y}) \qquad p_y = \frac{\partial T}{\partial \dot{y}} = 3m\dot{y} + m(\dot{x} + \dot{y})$$

$$p_y = m(4\dot{y} + \dot{x})$$

$$m\dot{x} = p_x - m\dot{y}$$

$$p_y = m\left(4\dot{y} + \frac{p_x - m\dot{y}}{m}\right) = p_x + 3m\dot{y}$$

$$\boxed{\dot{y} = \frac{p_y - p_x}{3m}}$$

$$m\dot{x} = p_x - \frac{p_y}{3} + \frac{p_x}{3} = \frac{4p_x - p_y}{3}$$

$$\boxed{\dot{x} = \frac{4p_x - p_y}{3m}}$$

$$T = \frac{3}{2}m\left(\frac{p_y - p_x}{3m}\right)^2 + \frac{m}{2}\left(\frac{4p_x - p_y}{3m} + \frac{p_y - p_x}{3m}\right)^2$$

$$= \frac{1}{6m}\left(p_y - p_x\right)^2 + \frac{1}{2m}\,p_x^2$$

$$= \frac{1}{6m}\left(p_y^2 + p_x^2 - 2p_y p_x + 3p_x^2\right)$$

$$= \frac{1}{6m}\left(4p_x^2 + p_y^2 - 2p_y p_x\right)$$

$$\boxed{\mathcal{H} = \frac{1}{6m}\left(4p_x^2 + p_y^2 - 2p_y p_x\right) + \frac{1}{2}kx^2} \qquad \frac{\partial \mathcal{H}}{\partial y} = 0$$

C) Hamilton's equations are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p_x} = \frac{1}{3m}\left(4p_x - p_y\right), \quad \dot{y} = \frac{\partial \mathcal{H}}{\partial p_y} = \frac{1}{3m}\left(p_y - p_x\right)$$

$$\dot{p}_x = -\frac{\partial \mathcal{H}}{\partial x} = -kx \qquad \dot{p}_y = -\frac{\partial \mathcal{H}}{\partial y} = 0$$

$\boxed{p_y \text{ is conserved}}$

$\longrightarrow$ since $\dot{x}(0) = \dot{y}(0) = 0 \longrightarrow p_y(0) = 0 \longrightarrow \boxed{p_y(t) = 0}$

Consequently

$$\dot{y} = -\frac{p_x}{3m} \qquad \dot{x} = \frac{4p_x}{3m} \qquad p_x = \frac{3}{4}m\dot{x}$$

$$\dot{p}_x = \frac{3}{4} m \ddot{x} = -kx \qquad \boxed{\ddot{x} = -\frac{4k}{3m} x} \qquad \omega = \sqrt{\frac{4k}{3m}}$$

$$\boxed{x = x_0 \cos(\omega t)} \qquad x(0) = x_0 \checkmark$$

$$\dot{y} = -\frac{1}{4} \dot{x} \qquad y = -\frac{1}{4} x + y_0$$

$$\boxed{y = -\frac{x_0}{4} \cos(\omega t) + y_0 + \frac{x_0}{4}} \qquad y(0) = y_0 \checkmark$$