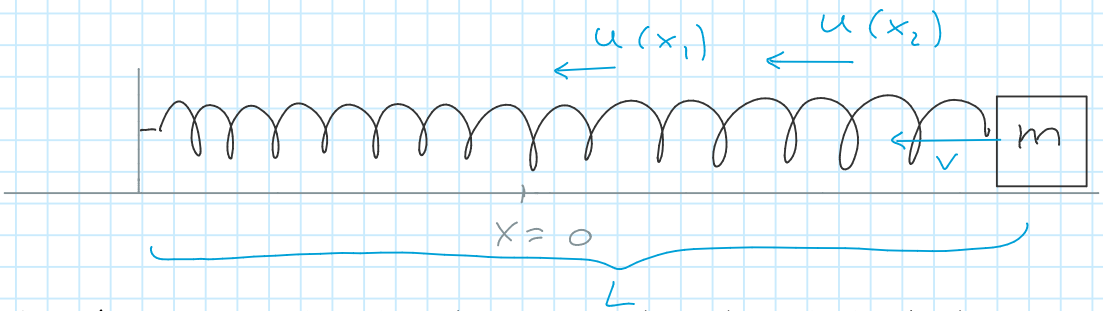
Massive spring

Monday, November 18, 2019 8:08 AM

Taylor, problem 13.6

In discussing the oscillations of a cart on the end of a spring, one almost always ignores the mass of the spring. Set up the Hamiltonian for the cart of mass m on a spring of force constant k whose mass M is non negligible. Use the extension of the spring x as the generalized coordinate. Solve Hamilton's equations

The problem here is to find the kinetic energy of the spring, since each point along the spring has a different velocity u



The key observation is that the velocity grows linearly with the displacement

$$u = v\,\frac{x}{L}$$

velocity at $x = L$

total length of the spring

Therefore the kinetic energy of the spring can be written as

$$T = \frac{1}{2}\int dm\; u^2(x) \qquad dm = \rho\, dx = \frac{M}{L}\,dx$$

$$T = \frac{1}{2}\int_0^L \frac{M}{L}\,dx\,\frac{v^2}{L^2}\,x^2 = \frac{M}{2L^3}v^2\int_0^L x^2\,dx = \frac{M}{6}v^2$$

The kinetic energy of the print does not depend on the length of the spring.
The potential energy is the same as for the massless spring, therefore the Hamiltonian is

$$\mathcal{L} = \frac{1}{2}\left(m + \frac{M}{3}\right)\dot{x}^2 - \frac{1}{2}kx^2$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = \left(m + \frac{M}{3}\right)\dot{x} \qquad \dot{x} = \frac{p}{m + \frac{M}{3}}$$

$$\mathcal{H} = p\dot{q} - \mathcal{L} = \frac{1}{2}\frac{p^2}{m + \frac{M}{3}} + \frac{1}{2}kx^2$$

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m + \frac{M}{3}} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = -kx$$

$$\left(m + \frac{M}{3}\right)\ddot{x} = -kx$$

$$\boxed{\ddot{x} = -\frac{k}{m + \frac{M}{3}}x}$$

SIMPLE HARMONIC MOTION WITH FREQUENCY

$$\omega = \sqrt{\frac{k}{m + \frac{M}{3}}}$$