Two masses connected by a spring and thrown upward

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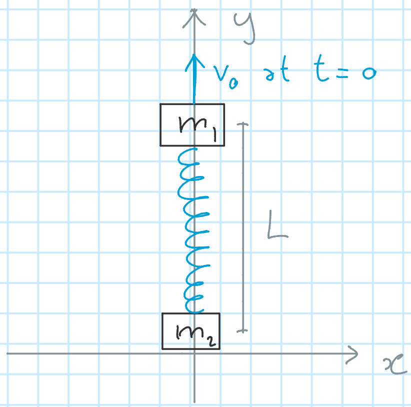
Taylor, problem 8.3

Two masses m1 and m2 are joined by a massless spring of natural length L and force constant k. Initially m2 is resting on a table and I am holding m1 vertically above m2 at height L. At the time t=0, I project m1 vertically upward with initial velocity v0 find the position of the two masses at any subsequent time t (before either mass returns to the table).



By using the results and notations of problem 8.2 one can write

$$\mathcal{L} = \frac{1}{2} M \dot{Y}^2 + \frac{1}{2}\mu \dot{y}^2 - \frac{1}{2}k(y-L)^2 - gMY$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{Y}} - \frac{\partial \mathcal{L}}{\partial Y} = 0 = M\ddot{Y} + gM \rightarrow \ddot{Y} = -g$$

One at this point needs to determine the initial conditions for the center of mass

$$Y = \frac{m_1 y_1 + m_2 y_2}{M} \xrightarrow{t=0} \frac{m_1 L}{M}$$

$$\dot{Y} = \frac{m_1 \dot{y}_1 + m_2 \dot{y}_2}{M} \xrightarrow{t=0} \frac{m_1 v_0}{M}$$

$$\boxed{Y = -\frac{1}{2} g t^2 + \frac{m_1}{M} v_0 t + \frac{m_1}{M} L}$$

Lagrange equation for the relative position is instead

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} - \frac{\partial \mathcal{L}}{\partial y} = 0 = \mu \ddot{y} + K(y-L) \qquad \ddot{y} = -\frac{K}{\mu}(y-L)$$

$$y = y_1 - y_2 \xrightarrow{t=0} y = L$$

$$\dot{y} = \dot{y}_1 - \dot{y}_2 \xrightarrow{t=0} \dot{y} = v_0$$

$$w \equiv y - L \longrightarrow \ddot{w} = -\frac{K}{\mu} w \qquad \omega \equiv \sqrt{\frac{K}{\mu}}$$

$$w = A \sin(\omega t) + B \cos(\omega t)$$

$$y = A \sin(\omega t) + B \cos(\omega t) + L$$

Fix the constants A and B for the given initial conditions

$$y(t=0) = L \longrightarrow B = 0$$

$$\dot{y} = A\omega \cos(\omega t)$$

$$\dot{y}(t=0) = v_0 \longrightarrow A\omega = v_0 \longrightarrow A = \frac{v_0}{\omega}$$

$$\boxed{y = \frac{v_0}{\omega}\sin(\omega t) + L}$$

At this stage one can write down the results for the coordinates of the two masses

$$y_1 = Y + \frac{m_2}{M} y \qquad y_2 = Y - \frac{m_1}{M} y$$

$$y_1 = -\frac{1}{2} g t^2 + \frac{m_1}{M} v_0 t + \frac{m_1}{M} L + \frac{m_2}{M}\frac{v_0}{\omega}\sin(\omega t) + \frac{m_2}{M} L$$

$$y_2 = -\frac{1}{2} g t^2 + \frac{m_1}{M} v_0 t + \cancel{\frac{m_1}{M} L} - \frac{m_1}{M}\frac{v_0}{\omega}\sin(\omega t) - \cancel{\frac{m_1}{M} L}$$

$$y_2 = -\frac{1}{2} g t^2 + \frac{m_1}{M} v_0 t + \cancel{\frac{m_1}{M} L} - \frac{m_1}{M} \frac{v_0}{\omega} \sin(\omega t) - \cancel{\frac{m_1}{M} L}$$

$$y_1 = -\frac{g}{2} t^2 + \frac{v_0}{M} \left(m_1 t + m_2 \frac{1}{\omega} \sin(\omega t) \right) + L$$

$$y_2 = -\frac{g}{2} t^2 + \frac{v_0}{M} m_1 \left(t - \frac{1}{\omega} \sin(\omega t) \right)$$

$y_1 (t=0) = L$ $\checkmark$ $y_2 (t=0) = 0$ $\checkmark$

$$\dot{y}_1 = -g t + \frac{m_1}{M} v_0 + \frac{m_2}{M} v_0 \cos(\omega t)$$

$y_1 (t=0) = v_0$ $\checkmark$

$$\dot{y}_2 = -g t + \frac{v_0}{M} m_1 - \frac{v_0 m_1}{M} \cos(\omega t)$$

$y_2 (t=0) = 0$ $\checkmark$