Two body problem in external gravitational field

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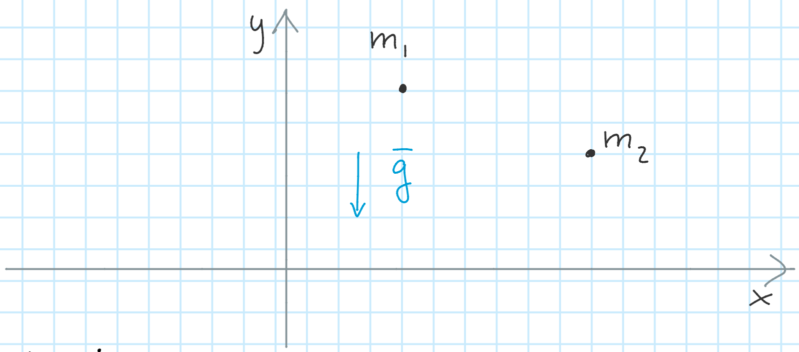
Two masses move in a uniform gravitational field grand interact via a potential U(r). Show that the Lagrangian can be decomposed in a Lagrangian for an object with the total mass of the system located in the center of mass and an object of reduced mass located at the relative position of the two objects. Write down Lagrange equations for the center of mass R and for the relative position r.



The Lagrangian of the system is

$$\mathcal{L} = \frac{1}{2} m_1 \dot{\bar{r}}_1^{\,2} + \frac{1}{2} m_2 \dot{\bar{r}}_2^{\,2} - U(r) - m_1 g y_1 - m_2 g y_2$$

The goal is now to rewrite the Lagrangian in terms of the variables

$$\bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{\underbrace{m_1 + m_2}_{\equiv M}} \qquad \text{CENTER OF MASS}$$

$$\bar{r} = \bar{r}_1 - \bar{r}_2 \qquad \text{RELATIVE POSITION}$$

$$\bar{r}_1 = \bar{r} + \bar{r}_2$$

$$\bar{R} = \frac{(m_1 + m_2)\bar{r}_2 + m_1 \bar{r}}{M} = \bar{r}_2 + \frac{m_1}{M}\bar{r}$$

$$\boxed{\bar{r}_2 = \bar{R} - \frac{m_1}{M}\bar{r}}$$

$$\bar{r}_1 = \bar{R} - \frac{m_1}{M}\bar{r} + \bar{r} \longrightarrow \boxed{\bar{r}_1 = \bar{R} + \frac{m_2}{M}\bar{r}}$$

The Lagrangian then becomes

$$\mathcal{L} = \frac{1}{2} m_1 \left(\dot{\bar{R}}^2 + \frac{m_2}{M} \dot{\bar{r}}^2 \right) + \frac{1}{2} m_2 \left(\dot{\bar{R}} - \frac{m_1}{M} \dot{\bar{r}} \right)^2 - U(r)$$

$$- m_1 g \left(Y + \frac{m_2}{M} y \right) - m_2 g \left(Y - \frac{m_1}{M} y \right)$$

$$\rightarrow \quad \bar{R} \equiv X\hat{i} + Y\hat{j} + Z\hat{k} \qquad \bar{r} = x\hat{i} + y\hat{j} + z\hat{k} \qquad \mu = \frac{m_1 m_2}{M}$$

$$\mathcal{L} = \underbrace{\frac{1}{2} M \dot{\bar{R}}^2 - M g Y}_{\mathcal{L}_{CM}} + \underbrace{\frac{1}{2} \mu \dot{\bar{r}}^2 - U(r)}_{\mathcal{L}_{rel}}$$