# Ignorable coordinates

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One aspect of the Hamiltonian formalism that shows some advantages with respect to the Lagrangian formalism is the handling of ignorable coordinates. These type of coordinates were defined as the generalized coordinates that do not appear explicitly in the Lagrangian:

$$\frac{\partial \mathcal{L}}{\partial q_i} = 0 \longrightarrow q_i = \text{ignorable coordinate}$$

If a given q is ignorable the corresponding generalized momentum is constant

$$q_i \text{ is ignorable} \longrightarrow \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0 \longrightarrow \dot{p}_i = 0 \longrightarrow p_i = \text{const}$$

Of course, this fact emerges also from the Hamiltonian formalism, since

$$\boxed{\frac{\partial \mathcal{L}}{\partial q_i} = -\frac{\partial \mathcal{H}}{\partial q_i}}$$

Proof:

$$\mathcal{H} = \sum_{j=1}^{n} p_j \dot{q}_j - \mathcal{L}(\{q_k, \dot{q}_k(q,p)\}, t)$$

$$\frac{\partial \mathcal{H}}{\partial q_i} = \cancel{\sum_{j=1}^{n} p_j \frac{\partial \dot{q}_j}{\partial q_i}} - \left( \frac{\partial \mathcal{L}}{\partial q_i} + \cancel{\sum_{k=1}^{n} \underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}_k}}_{= p_k} \frac{\partial \dot{q}_k}{\partial q_i}} \right)$$

$$= -\frac{\partial \mathcal{L}}{\partial q_i} \quad \checkmark$$

Consequently, for an ignorable coordinate

$$\dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} = 0 \longrightarrow p_i = \text{const}$$

If an Hamiltonian has an ignorable coordinate, it is straightforward to reduce the problem to a problem with one less degree of freedom. As an example one can consider an Hamiltonian that involves two generalized coordinates, one of which is ignorable.

$$\mathcal{H}(q_1, p_1, p_2) \longrightarrow \frac{\partial \mathcal{H}}{\partial q_2} = 0 \rightarrow$$ $q_2$ is ignorable

An Hamiltonian of type is for example the Hamiltonian of a particle moving on a plane under the action of a central force

$$\mathcal{H}(r, p_r, p_\phi) = \frac{1}{2m}\left(p_r^2 + \frac{p_\phi^2}{r^2}\right) + U(r)$$

$\hookrightarrow$ $\phi$ is ignorable

Since p_2 is constant, the problem immediately reduces to a problem with only one degree of freedom

$$p_2 \equiv k \longrightarrow \mathcal{H}(q_1, p_1, k) \rightarrow$$ 1 generalized coordinate

$$p_\phi \equiv \ell \longrightarrow \mathcal{H}(r, p_r, \ell)$$ $\ell$ now a parameter

This simplification is not immediately seen in the Lagrangian formalism, since the generalized velocity can still be time dependent even when the generalized momentum is a constant of motion. In our example

$$p_\phi = m r^2 \dot{\phi}$$

$p_\phi$: constant of motion

$r$: can change in time

$\dot{\phi}$: can change in time