Atwood's machine

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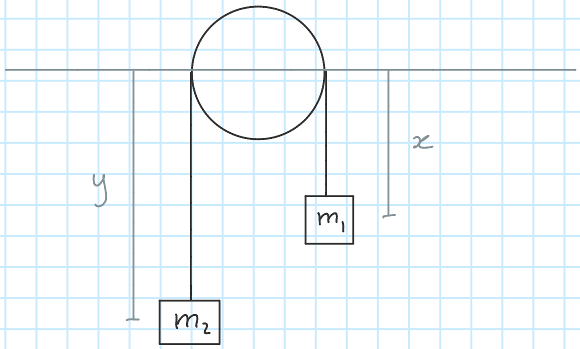
Atwood's machine is a system that can be easily treated with the tools of Hamiltonian mechanics.



$$T = \frac{1}{2}(m_1 + m_2)\dot{x}^2 \qquad U = -m_1 g x - m_2 g y + \text{const.}$$

but $x + y = \ell \quad y = \ell - x$

and one can choose $\text{const} = m_2 g \ell$, so that

$$U = -(m_1 - m_2) g x$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2)\dot{x}^2 + (m_1 - m_2) g x$$

$$p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = (m_1 + m_2)\dot{x}$$

$$\mathcal{H} = T + U = \frac{1}{2}\frac{p^2}{(m_1 + m_2)} - (m_1 - m_2) g x$$

Therefore, Hamilton's equations are

$$\dot{x} = \frac{\partial \mathcal{H}}{\partial p} = \frac{p}{m_1 + m_2} \qquad \dot{p} = -\frac{\partial \mathcal{H}}{\partial x} = (m_1 - m_2) g$$

By combining the two equations one finds the usual expression for the acceleration

$$(m_1 + m_2)\ddot{x} = (m_1 - m_2) g$$

$$\boxed{\ddot{x} = \frac{m_1 - m_2}{m_1 + m_2} g}$$