Introduction to Hamiltonian mechanics

Wednesday, October 9, 2019 10:49 AM

Besides Newtonian mechanics and Lagrangian mechanics, a third equivalent formulation of mechanics exists, and it is called Hamiltonian mechanics after William Rowan Hamilton (1805-1865), an irish mathematician that gave its final formulation.

At the classical level, Hamiltonian and Lagrangian mechanics share several of the same advantages. Hamiltonian mechanics has the further advantage that the fundamental quantity, the Hamiltonian, coincides for most systems with the total energy of the system, which is a quantity with a direct physical interpretation.

It is in quantum mechanics that the Hamiltonian formulation shows advantages with respect to the Lagrangian formulation (Schroedinger involves the Hamiltonian of the system). However, in quantum field theory, one can use either Hamiltonian or Lagrangians, and indeed is the approach based on Lagrangians (that enter in the path integral) that turns out to be more useful in most cases. In any case, clearly both formulations are useful and needed in the study of physics.

One can approach Hamiltonian mechanics starting from Lagrangian mechanics: In order to discuss mechanics in Lagrangian terms one needs to write the Lagrangian of the system

$$\mathcal{L} = \mathcal{L}(q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n, t) = T - U$$

$\{q_1, \dots, q_n\}$ → THE SET OF ALL OF THESE POINTS DEFINE THE CONFIGURATION SPACE

$\{q_1, \dots, q_n, \dot{q}_1, \dots, \dot{q}_n\}$ → DEFINES THE STATE SPACE (2n coordinates)

Now we introduce a letter to indicate the generalized momenta

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$$

GENERALIZED MOMENTA

The generalized momentum are also referred to as canonical momentum or conjugate momentum.

The Hamiltonian is then defined as

$$\mathcal{H} = \sum_{i=1}^{n} p_i \dot{q}_i - \mathcal{L}$$

HAMILTONIAN

We already discussed how the Hamiltonian is the energy of the system, provided that the generalized coordinates are natural, i.e. the relation between the generalized coordinates and the underlying Cartesian coordinates does not depend on time.

In Hamiltonian mechanics the systems evolves by moving from point to point in the phase space

$$\{ \underbrace{q_1, \dots, q_n}_{\text{generalized coordinates}}, \underbrace{p_1, \dots, p_n}_{\text{generalized momenta}} \} \longrightarrow \text{PHASE SPACE (2n coordinates)}$$

The Hamiltonian approach allows one to derive the equations of motion of the system starting from the Hamiltonian of the system. As for the Lagrangian case, also the Hamiltonian approach is developed to describe conservative systems, so that we only consider problems in which friction forces do not play a role, as it is often the case in astrophysics and microscopic physics.