Relation between energy and eccentricity

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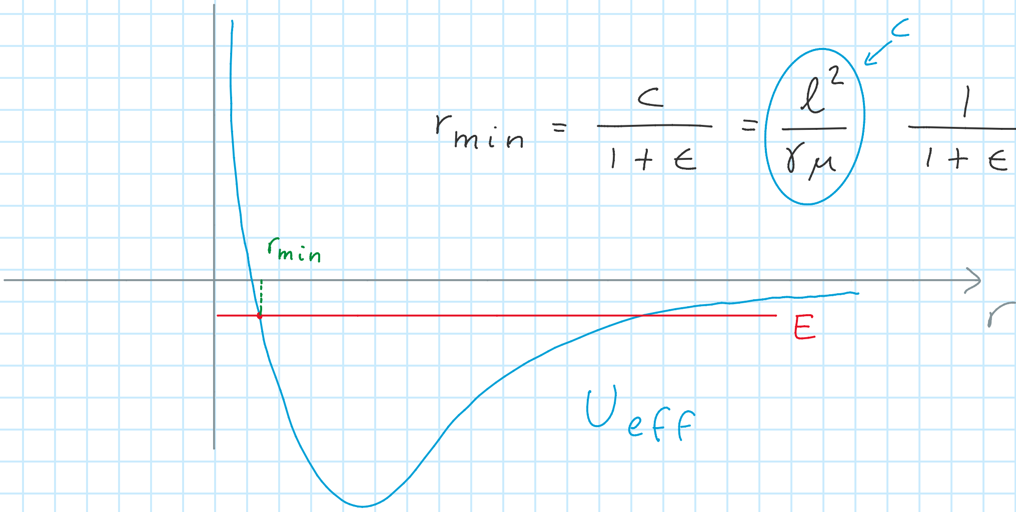
It is useful to find an equation that relates the energy of the system, the angular momentum and the eccentricity of the orbit. This can be done easily by remembering that at the aphelion, the point of closest approach between the two orbiting masses, the energy is equal to the effective potential. Notice that is true for both bounded and unbounded orbits.

$$r_{min} = \frac{c}{1+\epsilon} = \underbrace{\frac{\ell^2}{\gamma\mu}}_{c} \frac{1}{1+\epsilon}$$



$$E = U_{eff}(r_{min}) = -\frac{\gamma}{r_{min}} + \frac{\ell^2}{2\mu r_{min}^2}$$

$$= \frac{1}{2 r_{min}} \left(\frac{\ell^2}{\mu r_{min}} - 2\gamma \right)$$

$$= \frac{\gamma\mu(1+\epsilon)}{2\ell^2} \left(\gamma(1+\epsilon) - 2\gamma \right)$$

$$= \frac{\gamma^2\mu(1+\epsilon)}{2\ell^2}(\epsilon - 1) = \frac{\gamma^2\mu}{2\ell^2}(\epsilon^2 - 1)$$

From the equation above one immediately concludes that $\varepsilon<1$ correspond to negative energy and bounded orbits, while $\varepsilon>1$ corresponds to positive energy and unbounded orbits. The smallest possible energy for a given angular momentum is attained when $\varepsilon=0$, i.e. for the circular orbit.