Orbit equation

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At this stage we reduced the two body problem to a one dimensional problem, and we obtained a differential equation for r, that, if solved, allows one to obtain r(t). However, one is interested also in finding the trajectory of the particle of mass μ in the center of mass rest frame, i.e. one is interested in finding $r(\phi)$.

It turn out that it is more convenient to change variables and trade r with u =1/r

$$\mu \ddot{r} = -\frac{dU}{dr} + \frac{\ell^2}{\mu r^3}$$

$$\mu \frac{d^2}{dt^2}\left(\frac{1}{u}\right) = -\frac{dU}{du}\frac{du}{dr} + \frac{u^3\ell^2}{\mu}$$

$$\frac{dr}{du} = \frac{d}{du}\left(\frac{1}{u}\right) = -\frac{1}{u^2}$$

$$\hookrightarrow \frac{du}{dr} = -u^2$$

$$\mu \frac{d}{dt}\left(-\frac{1}{u^2}\frac{du}{dt}\right) = +\frac{dU}{du}u^2 + \frac{u^3\ell^2}{\mu}$$

$$\mu\left[\frac{2}{u^3}\left(\frac{du}{dt}\right)^2 - \frac{1}{u^2}\frac{d^2u}{dt^2}\right] = \frac{dU}{du}u^2 + \frac{u^3\ell^2}{\mu}$$

$$\frac{2\mu}{u}\dot{u}^2 - \mu\ddot{u} = \frac{dU}{du}u^4 + \frac{u^5\ell^2}{\mu}$$

In addition one can trade the time derivative with a derivative with respect to Φ

$$\frac{d}{dt} = \frac{d\phi}{dt}\frac{d}{d\phi} = \dot{\phi}\frac{d}{d\phi} = \frac{\ell}{\mu r^2}\frac{d}{d\phi} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}$$

$$\dot{u} = \frac{\ell u^2}{\mu}\frac{du}{d\phi} \qquad \ddot{u} = \frac{\ell u^2}{\mu}\frac{d}{d\phi}\left(\frac{\ell u^2}{\mu}\frac{du}{d\phi}\right) = \frac{\ell^2 u^2}{\mu^2}\left[2u\left(\frac{du}{d\phi}\right)^2 + u^2\frac{d^2u}{d\phi^2}\right]$$

The equation then becomes

$$\frac{2\cancel{\mu}}{\cancel{u}}\frac{\ell^2 u^{\cancel{4}\,3}}{\mu^{\cancel{2}}}\cancel{\left(\frac{du}{d\phi}\right)^2} - \cancel{\mu}\frac{\ell^2 u^2}{\mu^{\cancel{2}}}\left[2u\cancel{\left(\frac{du}{d\phi}\right)^2} + u^2\frac{d^2u}{d\phi^2}\right] = \frac{dU}{du}u^4 + \frac{u^5\ell^2}{\mu}$$

$$-\frac{\ell^2 u^4}{\mu}\frac{d^2u}{d\phi^2} = \frac{dU}{du}u^4 + \frac{u^5\ell^2}{\mu}$$

$$\boxed{\frac{d^2u}{d\phi^2} = -\frac{\mu}{\ell^2}\frac{dU}{du} - u}$$

This can further be rewritten in terms of the force, if desired

$$F = -\frac{dU}{dr} = -\frac{dU}{du}\frac{du}{dr} = u^2\frac{dU}{du}$$

$$\boxed{\frac{d^2u}{d\phi^2} = -u - \frac{\mu}{\ell^2 u^2}F} \longrightarrow \text{SOLVE TO FIND } u(\phi)$$

Up to this point that discussion was kept general and no specific central conservative force was used in the derivation. As a first application of the method, one can study the equation above in absence of a force, i.e. for a free particle.

Radial equation for a free particle

In the case of a free particle the equation reduces to

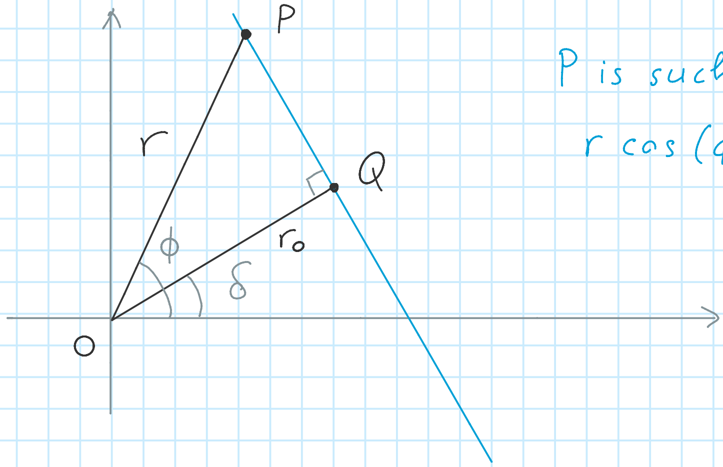
$$\frac{d^2u}{d\phi^2} = -u(\phi)$$

This is an equation that is formally identical to the equation of the simple harmonic oscillator. Therefore the general solution is

$$u(\phi) = A\cos(\phi - \delta) \qquad r_0 \equiv \frac{1}{A} \quad \text{rename const.}$$

$$r(\phi) = \frac{1}{u(\phi)} = \frac{1}{A\cos(\phi-\delta)} \equiv \frac{r_0}{\cos(\phi-\delta)}$$

As expected, the equation above describes a straight line in polar coordinates



P is such that $r \cos(\phi - \delta) = r_0$

Kepler orbits

Now we want to find the orbit in the case of a central force that decreases with distance with the inverse square law. This is of course relevant for the case of planetary orbits around the sun, as well as for the orbits of satellites around planets. One can write down the force as follows

$$F = G \frac{m_1 m_2}{r^2} \equiv \frac{\gamma}{r^2}$$

$\boxed{\gamma \equiv G m_1 m_2}$

FORCE CONSTANT

The relevant differential equation to be solved in this case becomes

$$\boxed{u''(\phi) = -u(\phi) + \frac{\gamma \mu}{\ell^2}}$$

The last term in the differential equation is in this case a constant. One can then replace the function u with the function w

$$w(\phi) \equiv u(\phi) - \gamma \frac{\mu}{\ell^2} \longrightarrow w''(\phi) = -w(\phi)$$

The general solution of the equation is therefore

$$\boxed{w(\phi) = A \cos(\phi - \delta)}$$

One can immediately fix the constant δ to zero by choosing appropriately the direction of the x axis, with respect to which the angle ϕ is measured. Consequently

$$u(\phi) = \frac{\gamma\mu}{\ell^2} + A\cos\phi = \frac{\gamma\mu}{\ell^2}\left(1 + \underbrace{\epsilon}_{\equiv \frac{A\ell^2}{\gamma\mu}}\cos\phi\right)$$

$$= \underbrace{\frac{1}{c}}_{c \equiv \frac{\ell^2}{\gamma\mu}}(1 + \epsilon\cos\phi)$$

it has the dimensions of length

Therefore

$$u(\phi) = \frac{1}{r(\phi)} = \frac{1}{c}(1 + \epsilon\cos\phi)$$

$$\boxed{r(\phi) = \frac{c}{1 + \epsilon\cos\phi}}$$

ORBIT EQUATION

The behavior of the solution depends on the value of the parameter ε. Indeed if $\varepsilon<1$, the denominator in the orbit equation cannot vanish. Consequently r is always finite and the orbit is said to be bound. In that case, the largest and smallest values of r are

$$\underbrace{r_{min}}_{\text{perihelion}} = \frac{c}{1+\epsilon} \qquad \underbrace{r_{max}}_{\text{aphelion}} = \frac{c}{1-\epsilon}$$

Since r is a periodic function of ϕ of period 2π, the orbit is not only bound but it is closed. The orbit is indeed an ellipse

$$x = r\cos\phi \qquad y = r\sin\phi$$

$$r + \epsilon r \cos\phi = c$$

$$\sqrt{x^2+y^2} + \epsilon x = c$$

$$\sqrt{x^2+y^2} = c - \epsilon x$$

$$x^2 + y^2 = c^2 + \epsilon^2 x^2 - 2\epsilon c x$$

$$x^2 + 2\epsilon c x - \epsilon^2 x^2 + y^2 = c^2$$

$$(1-\epsilon^2)x^2 + 2\epsilon c x + y^2 = c^2$$

$$x^2 + 2\frac{\epsilon c}{1-\epsilon^2}x + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$\Big(x + \underbrace{\frac{\epsilon c}{1-\epsilon^2}}_{\equiv d}\Big)^2 - \frac{\epsilon^2 c^2}{(1-\epsilon^2)^2} + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}$$

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}\left(1 + \frac{\epsilon^2}{1-\epsilon^2}\right)$$

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = \frac{c^2}{1-\epsilon^2}\left(\frac{1-\epsilon^2+\epsilon^2}{1-\epsilon^2}\right)$$

$$(x+d)^2 + \frac{y^2}{1-\epsilon^2} = \underbrace{\frac{c^2}{(1-\epsilon^2)^2}}_{\equiv a^2}$$

observe

$d = \epsilon a$

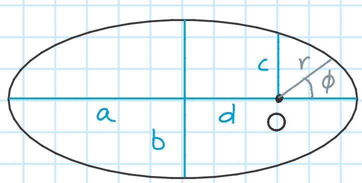
$$\frac{(x+d)^2}{a^2} + \underbrace{\frac{1-\epsilon^2}{c^2}}_{\equiv \frac{1}{b^2}} y^2 = 1 \longrightarrow \boxed{\frac{(x+d)^2}{a^2} + \frac{y^2}{b^2} = 1}$$

ELLIPSE

a = semimajor axis

b = semiminor axis

ϵ = eccentricity

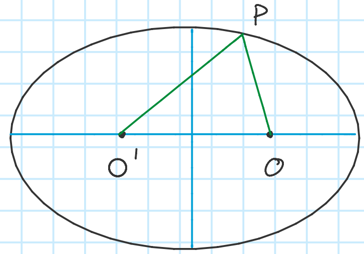


$$\boxed{\frac{b}{a} = \sqrt{1-\epsilon^2}}$$

if $\epsilon = 0 \rightarrow a = b$

$\hookrightarrow$ circle

Now let's prove that the origin O is located in one of the focuses of the ellipse. Indeed, if O is a focus, for any point in the ellipse the sum of the distances of the point from O and from the point symmetric to it with respect to the center of the ellipse should be constant (independent of ϕ)



$$\overline{OP} + \overline{O'P} = \sqrt{x^2+y^2} + \sqrt{(x+2d)^2+y^2}$$

$$= r + \sqrt{x^2+y^2+4dx+4d^2}$$

$$= r + \sqrt{r^2+4dr\cos\phi+4d^2}$$

$$r = \frac{c}{1+\epsilon\cos\phi} \longrightarrow \cos\phi = \frac{c-r}{r\epsilon}$$

argument of the square root

$$r^2 + 4\cancel{r}\frac{\cancel{\epsilon} c}{1-\epsilon^2}\,\frac{c-r}{\cancel{r\epsilon}} + 4\frac{\epsilon^2c^2}{(1-\epsilon^2)^2} =$$

$$\frac{1}{(1-\epsilon^2)^2}\left[(1-\epsilon^2)^2r^2 + 4c(c-r)(1-\epsilon^2) + 4\epsilon^2c^2\right]$$

$$= \frac{1}{(1-\epsilon^2)^2}\left[(1+\epsilon^4-2\epsilon^2)r^2 + 4c\left(c-c\epsilon^2-r+r\epsilon^2\right)+4\epsilon^2c^2\right]$$

$$= \frac{1}{(1-\epsilon^2)^2}\left[r^2+r^2\epsilon^4-2\epsilon^2r^2+4c^2-\cancel{4c^2\epsilon^2}-4cr+4cr\epsilon^2+\cancel{4\epsilon^2c^2}\right]$$

$$= \frac{1}{(1-\epsilon^2)^2}\left[r^2 + r^2\epsilon^4 + 4c^2 - 2\epsilon^2 r^2 - 4cr + 4cr\epsilon^2\right]$$

$$= \frac{1}{(1-\epsilon^2)^2}\left(-r + r\epsilon^2 + 2c\right)^2 = \frac{1}{(1-\epsilon^2)^2}\left[2c - (1-\epsilon^2)r\right]^2$$

There fore

$$\overline{OP} + \overline{O'P} = r + \frac{1}{(1-\epsilon^2)}\left[2c - (1-\epsilon^2)r\right]$$

$$= \frac{1}{(1-\epsilon^2)}\left[\cancel{(1-\epsilon^2) r} + 2c - \cancel{(1-\epsilon^2) r}\right] = \frac{2c}{1-\epsilon^2} = 2a$$

It does not depend on r or ϕ, q.e.d.