# Isochronous oscillations

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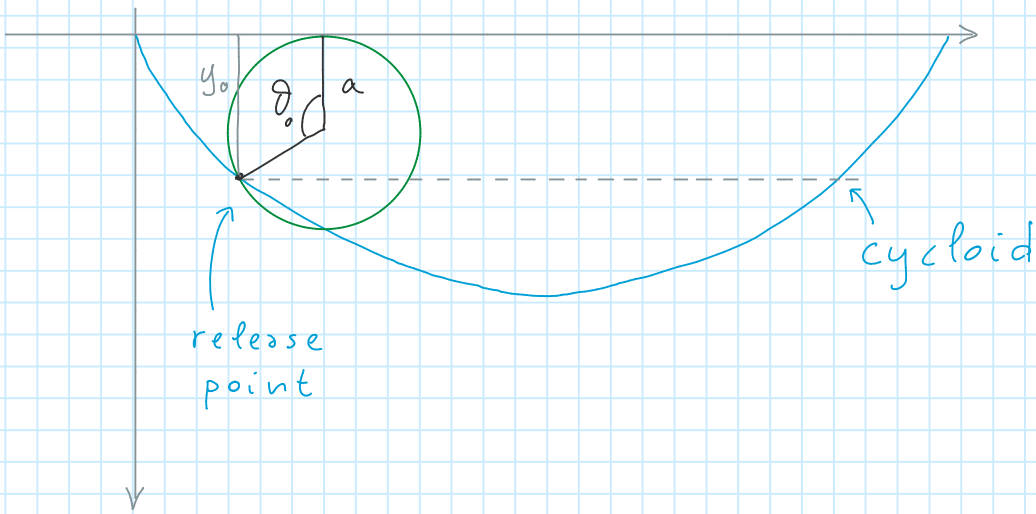
From Taylor chapter 6 problem 25

Consider a bowl shaped like a cycloid. Show that the oscillation period of a marble released at ANY point in the bowl is always the same, irrespectively from how far the release point is from the bottom of the bowl. The period of the oscillation is

$$T = 4\pi\sqrt{\frac{a}{g}}$$




The points on the cycloid satisfy the equations

$$x = a(\vartheta - \sin\vartheta) \qquad y = a(1 - \cos\vartheta)$$

The total energy of the marble is

$$E = \frac{1}{2}mv^2 - mgy + c$$

Since the marble is released from rest,

$$y = y_0 \longrightarrow v = 0 \qquad E = -mgy_0 + c \longrightarrow c = E + mgy_0$$

$$E = \frac{1}{2} m v^2 - m g (y - y_0) + E$$

$$\frac{1}{2} m v^2 - m g (y - y_0) = 0 \longrightarrow v = \sqrt{2 g (y - y_0)}$$

The infinitesimal distance ds traveled in the infinitesimal time dt is

$$ds = v\, dt \qquad dt = \frac{ds}{v}$$

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (x')^2}\, dy$$

The oscillation period is therefore

$$\frac{T}{4} = \int \frac{ds}{v} = \int_{y_0}^{2a} \frac{\sqrt{1 + (x')^2}}{\sqrt{2g (y - y_0)}}\, dy$$

But we already know that the trajectory is a cycloid, therefore

$$\frac{dx}{dy} = \frac{dx}{d\vartheta} \frac{d\vartheta}{dy} = \frac{a(1 - \cos\vartheta)}{a \sin\vartheta} = \frac{1 - \cos\vartheta}{\sqrt{1 - \cos^2\vartheta}}$$

$$= \sqrt{\frac{1 - \cos\vartheta}{1 + \cos\vartheta}}$$

$$dy = a \sin\vartheta\, d\vartheta$$

Therefore

$$\frac{T}{4} = \frac{1}{\sqrt{2ga}} \int_{\vartheta_0}^{\pi} \sqrt{1 + \frac{1 - \cos\vartheta}{1 + \cos\vartheta}} \frac{a \sin\vartheta\, d\vartheta}{\sqrt{\cos\vartheta_0 - \cos\vartheta}}$$

$$= \sqrt{\frac{a}{g}} \int_{\vartheta_0}^{\pi} \frac{\sqrt{1 - \cos^2\vartheta}\; d\vartheta}{\sqrt{1 + \cos\vartheta}\sqrt{\cos\vartheta_0 - \cos\vartheta}}$$

$$\frac{T}{4} = \sqrt{\frac{a}{g}} \int_{\vartheta_0}^{\pi} \sqrt{\frac{1-\cos\vartheta}{\cos\vartheta_0 - \cos\vartheta}}\, d\vartheta$$

Now one needs to evaluate the integral. This can be done through two changes of variables

$$\vartheta \equiv \pi - 2\alpha \qquad d\vartheta = -2\,d\alpha$$

$$\cos\vartheta = \cos(\pi - 2\alpha) = \underbrace{\cos\pi}_{-1} \cos 2\alpha + \cancelto{0}{\sin\pi} \sin 2\alpha$$

$$= -\cos(2\alpha)$$

$$\frac{T}{4} = -2\sqrt{\frac{a}{g}} \int_{\alpha_0}^{0} d\alpha \sqrt{\frac{1+\cos 2\alpha}{\cos 2\alpha - \cos 2\alpha_0}}$$

$$= +2\sqrt{\frac{a}{g}} \int_{0}^{\alpha_0} d\alpha \sqrt{\frac{1+\cos^2\alpha - \sin^2\alpha}{\cos^2\alpha - \sin^2\alpha - \cos^2\alpha_0 + \cos^2\alpha}}$$

$$= +2\sqrt{\frac{a}{g}} \int_{0}^{\alpha_0} d\alpha \sqrt{\frac{1-\sin^2\alpha}{\sin^2\alpha_0 - \sin^2\alpha}}$$

$$u \equiv \sin\alpha \qquad du = \cos\alpha\, d\alpha$$

$$\frac{T}{4} = 2\sqrt{\frac{a}{g}} \int_{0}^{u_0} du \frac{1}{\sqrt{u_0^2 - u^2}}$$

$$= -2\sqrt{\frac{a}{g}} \arctan\left(-\frac{u}{\sqrt{u_0^2 - u^2}}\right)\Bigg|_{0}^{u_0}$$

$$= -2\sqrt{\frac{a}{g}} \Big(\underbrace{\arctan(-\infty)}_{-\frac{\pi}{2}} - \cancelto{0}{\arctan 0}\Big)$$

$$= \pi\sqrt{\frac{a}{g}} \longrightarrow \boxed{T = 4\pi\sqrt{\frac{a}{g}}}$$

The result does not depend on $\vartheta_0$ !