Pendulum attached to a rotating disk

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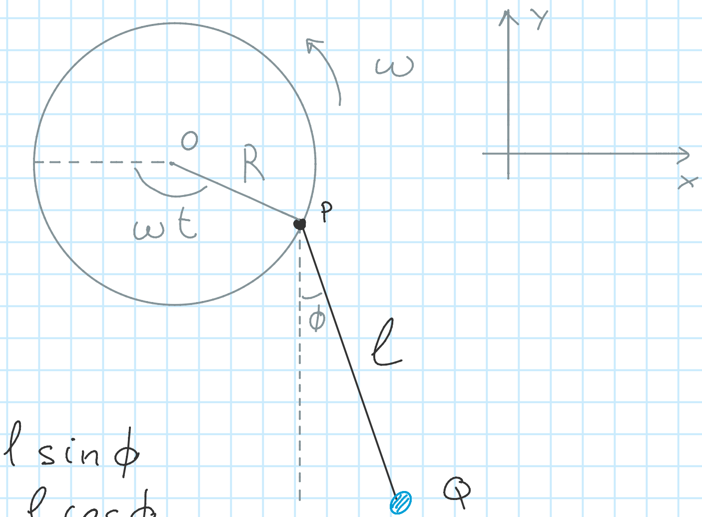
Problem 7.29 in Taylor's book

Suppose to have a pendulum attached to a point on the edge of a vertical wheel, rotating with fixed angular velocity ω. At $t = 0$, the fulcrum of the pendulum is level with the center of the wheel and to the left of it. Write down the Lagrangian of the system and the equation of motion for the angle Φ.



$$x_P = -R\cos(\omega t)$$
$$y_P = -R\sin(\omega t)$$

$$x \equiv x_Q = -R\cos(\omega t) + \ell\sin\phi$$
$$y \equiv y_Q = -R\sin(\omega t) - \ell\cos\phi$$

$$\dot{x} = \omega R\sin(\omega t) + \ell\dot{\phi}\cos\phi$$

$$\dot{y} = -\omega R\cos(\omega t) + \ell\dot{\phi}\sin\phi$$

$$\dot{x}^2 + \dot{y}^2 = \omega^2 R^2 \sin^2(\omega t) + \ell^2\dot{\phi}^2\cos^2\phi + 2\ell\omega R\dot{\phi}\sin(\omega t)\cos\phi$$
$$+ \omega^2 R^2\cos^2(\omega t) + \ell^2\dot{\phi}^2\sin^2\phi - 2\ell\omega R\dot{\phi}\cos(\omega t)\sin\phi$$
$$= \omega^2 R^2 + \ell^2\dot{\phi}^2 + 2\ell\omega R\dot{\phi}\sin(\omega t - \phi)$$

$$\mathcal{L} = \frac{m}{2}\left(\omega^2 R^2 + \ell^2\dot{\phi}^2 + 2\omega R\ell\dot{\phi}\sin(\omega t - \phi)\right)$$
$$- mg\left(-R\sin(\omega t) - \ell\cos\phi\right)$$

$$\frac{\partial\mathcal{L}}{\partial\dot{\phi}} = m\ell^2\dot{\phi} + m\omega R\ell\sin(\omega t - \phi)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = - m \omega R \ell \dot{\phi} \cos(\omega t - \phi) - m g \ell \sin\phi$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\phi}} - \frac{\partial \mathcal{L}}{\partial \phi} = m \ell^2 \ddot{\phi} + m \omega R \ell \cos(\omega t - \phi)(\omega - \cancel{\dot{\phi}})$$

$$+ m \omega R \ell \cancel{\dot{\phi} \cos(\omega t - \phi)} + m g \ell \sin\phi = 0$$

$$\boxed{\ell \ddot{\phi} + \omega^2 R \cos(\omega t - \phi) + g \sin\phi = 0}$$

if $\omega = 0$ one recovers the equation for the simple pendulum

$$\ddot{\phi} = - \frac{g}{\ell} \sin\phi$$