Soap on a tilting tray

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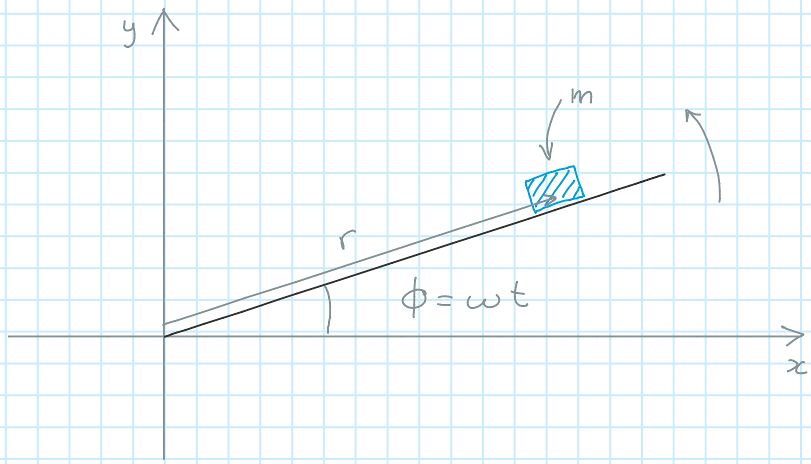
Problem 7.33 on Taylor

A bar of soap is at rest on a frictionless rectangular plate that rests on a horizontal table. At time $t = 0$, I start raising one edge of the plate so that the plate pivots about the opposite edge with constant angular velocity ω, and the soap starts to slide toward the downhill edge. Find the equation of motion for the coordinate r indicating the distance between the soap and the downhill edge. Solve the equation of motion.



$$\mathcal{L} = \frac{1}{2} m \left(\dot{r}^2 + r^2 \omega^2 \right) - m g r \sin(\omega t)$$

$$\frac{\partial \mathcal{L}}{\partial r} = m r \omega^2 - m g \sin(\omega t)$$

$$\frac{\partial \mathcal{L}}{\partial \dot{r}} = m \dot{r}$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{r}} - \frac{\partial \mathcal{L}}{\partial r} = m \ddot{r} - m r \omega^2 + m g \sin(\omega t) = 0$$

$$\boxed{\ddot{r} - r \omega^2 = - g \sin(\omega t)}$$

EQUATION OF MOTION

homogeneous solution

$$r_h = C_1 e^{-\omega t} + C_2 e^{+\omega t}$$

$$\dot{r}_h = \omega \left(-C_1 e^{-\omega t} + C_2 e^{\omega t} \right) \qquad \ddot{r}_h = +\omega^2 r_h$$

$$\ddot{r}_h - \omega^2 r_h = \emptyset$$

particular solution

$$r_p = A \sin(\omega t) \qquad \text{ANSATZ}$$

$$\dot{r}_p = A\omega \cos(\omega t) \qquad \ddot{r}_p = -A\omega^2 \sin(\omega t)$$

$$\ddot{r}_p - \omega^2 r_p + g \sin(\omega t) = 0$$

$$-A\omega^2 \sin(\omega t) - A\omega^2 \sin(\omega t) + g \sin(\omega t) = 0$$

$$-2A\omega^2 + g = 0 \qquad \boxed{A = \frac{g}{2\omega^2}}$$

$\boxed{\text{Solution}}$

$$r(t) = r_h(t) + r_p(t) = C_1 e^{-\omega t} + C_2 e^{+\omega t} + \frac{g}{2\omega^2} \sin(\omega t)$$

boundary conditions

$$r(0) = r_0 = C_1 + C_2$$

$$v(0) = 0 = \dot{r}(0) = \left(-\omega C_1 e^{-\omega t} + \omega C_2 e^{+\omega t} + \frac{g}{2\omega} \cos(\omega t) \right) \Big|_{t=0}$$

$$= \omega (C_2 - C_1) + \frac{g}{2\omega}$$

$$C_2 = C_1 - \frac{g}{2\omega^2}$$

$$r_0 = C_1 + C_2 = 2C_1 - \frac{g}{2\omega^2} \qquad C_1 = \frac{1}{2}\left(r_0 + \frac{g}{2\omega^2}\right)$$

$$C_2 = \frac{r_0}{2} + \frac{g}{4\omega^2} - \frac{g}{2\omega^2} = \frac{1}{2}\left(r_0 - \frac{g}{2\omega^2}\right)$$

Final solution

$$\boxed{r(t) = \frac{1}{2}\left(r_0 + \frac{g}{2\omega^2}\right) e^{-\omega t} + \frac{1}{2}\left(r_0 - \frac{g}{2\omega^2}\right) e^{\omega t} + \frac{g}{2\omega^2}\sin(\omega t)}$$