Center of mass frame

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We were able to rewrite the Lagrangian in the two body problem as the sum of a Lagrangian of a free particle of mass M located at the center of mass R and a particle of mass μ located at the position r

$$\mathcal{L} = \underbrace{\frac{1}{2} M \dot{\bar{R}}^2}_{\text{Free particle Lagrangian}} + \frac{1}{2}\mu \dot{\bar{r}}^2 - U(r)$$

$$\bar{R} \equiv X\hat{i} + Y\hat{j} + Z\hat{k}$$

The three components of the vector R are ignorable coordinates, which in turn implies that the total momentum of the system is conserved:

$$\frac{\partial \mathcal{L}}{\partial X} = \frac{\partial \mathcal{L}}{\partial Y} = \frac{\partial \mathcal{L}}{\partial Z} = 0 \longrightarrow \frac{d}{dt} M\dot{\bar{R}} = 0 \qquad \bar{P} = M\dot{\bar{R}} = \text{const.}$$

The equation of motion for r is the equation of a single particle of mass μ subject to the central potential U.

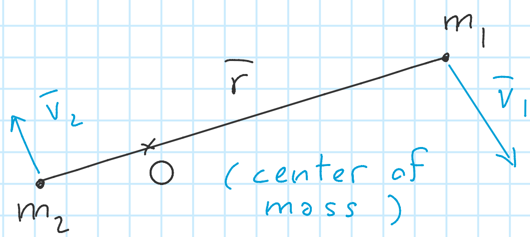
$$\mu \ddot{\bar{r}} = -\nabla U(r)$$

Center of mass frame

It is obviously convenient to work in the frame where
I) the total momentum of the system is zero
II) the center of mass of the system coincides with the frame origin
In this case the Lagrangian depends only on the vector r and on its first derivative.

$$\mathcal{L} = \frac{1}{2}\mu \dot{\bar{r}}^2 - U(r) \longrightarrow$$

The configuration of the system will look as follows



Notice that the velocities of the two objects are related

$$\bar{P} = 0 \quad \longrightarrow \quad m_1 \bar{v}_1 = -m_2 \bar{v}_2$$

$$m_1 \dot{\bar{r}}_1 = -m_2 \dot{\bar{r}}_2 \qquad \boxed{\dot{\bar{r}}_1 = -\frac{m_2}{m_1}\dot{\bar{r}}_2}$$

in addition

$$\bar{r} = \bar{r}_1 - \bar{r}_2 \qquad (\text{ but } r = r_1 + r_2)$$

$$\dot{\bar{r}}_1 = \dot{\bar{r}} + \dot{\bar{r}}_2 \qquad \dot{\bar{r}}_2 = \dot{\bar{r}}_1 - \dot{\bar{r}}$$

$$-\frac{m_2}{m_1}\dot{\bar{r}}_2 = \dot{\bar{r}} + \dot{\bar{r}}_2 \qquad \dot{\bar{r}}_2\left(1 + \frac{m_2}{m_1}\right) = -\dot{\bar{r}}$$

$$\boxed{\dot{\bar{r}}_2 = -\frac{m_1}{M}\dot{\bar{r}}} \; , \quad \boxed{\dot{\bar{r}}_1 = \frac{m_2}{M}\dot{\bar{r}}}$$

Of course, one can also visualize the system as a single particle of mass μ whose position is indicated by the vector r. If a mass (say m1) is much smaller than the other, the reduced mass will be very close to the smallest of the two masses, and the vector r1 will be almost identical to the vector r. Consequently one can interpret the Lagrangian as describing the motion of the small mass around the big mass. For example this description is useful when modeling the motion of the Earth around the Sun.

Angular momentum of the system

The angular momentum of the two body system is

$$\bar{L} = \bar{r}_1 \times \bar{p}_1 + \bar{r}_2 \times \bar{p}_2 = m_1 \bar{r}_1 \times \dot{\bar{r}}_1 + m_2 \bar{r}_2 \times \dot{\bar{r}}_2$$

insert

$$\bar{r}_1 = \frac{m_2}{M}\bar{r} \qquad \bar{r}_2 = -\frac{m_1}{M}\bar{r}$$

$$\bar{L} = \frac{m_2}{M}\bar{r} \times \frac{m_1 m_2}{M}\dot{\bar{r}} + \frac{m_1}{M}\bar{r} \times \frac{m_2 m_1}{M}\dot{\bar{r}}$$

$$= \frac{m_1 m_2}{M^2}\left(m_2 \bar{r} \times \dot{\bar{r}} + m_1 \bar{r} \times \dot{\bar{r}}\right) =$$

$$= \frac{m_1 m_2}{M} \bar{r} \times \dot{\bar{r}} = \bar{r} \times \mu \dot{\bar{r}}$$

This coincides with the angular momentum of a single particle of mass μ and coordinate r. Since the system is isolated there are no external torques and the angular momentum is conserved in magnitude and direction. The fact that the direction of the angular momentum does not change tells us that the motion stays on a plane. One can then identify the plane of the motion with the x-y plane. The two body problem is a two dimensional problem.

If one works in polar coordinates it is easy to see that the polar angle is an ignorable coordinate.

$$\mathcal{L} = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\phi}^2\right) - U(r)$$

$$\frac{\partial \mathcal{L}}{\partial \phi} = 0 \longrightarrow \frac{d}{dt}\mu r^2 \dot{\phi} = 0 \longrightarrow \underbrace{\mu r^2 \dot{\phi} = \text{const}}_{\text{cons. of } |\bar{L}| = \ell}$$

Equivalent one dimensional problem

The only relevant Lagrange equation is therefore the one for the generalized coordinate r. So the two body problem was reduced to a one dimensional problem for r

$$\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}} \longrightarrow \overbrace{\mu \ddot{r} - \mu r \dot{\phi}^2}^{\mu a_{\text{radial}}} + \frac{\partial U}{\partial r} = 0$$

Therefore one can interpret this equation as an equation for the radial acceleration of a particle of mass μ.