# The two body problem

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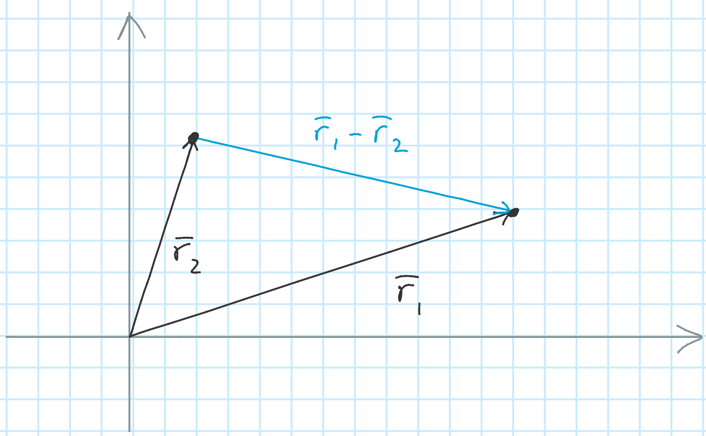
In many instances in physics one needs to describe a system formed by just two bodies that interact through central forces. In first approximation one can treat the system Sun-Earth as a two body problem. Similarly, one can treat the system Earth-Moon as a two body problem (obviously, a more detailed discussion of this system will need to account for the effect of the Sun gravitational force on both the Earth and the Moon).

Also the system electron proton (hydrogen atom), dominated by Coulomb's force can be treated as a two body problem, although in this case one needs to use quantum mechanics. Several aspects of the study of the two body problem in classical mechanics can be carried over to quantum mechanics.

In a generic frame of reference the situation will be the following




We consider the two objects as point like. The potential energy of the system will be

$$U(\bar{r}_1, \bar{r}_2) = -G \frac{m_1 m_2}{|\bar{r}_1 - \bar{r}_2|}$$

GRAVITATIONAL FORCE

(ex. SUN-EARTH)

$$U(\bar{r}_1, \bar{r}_2) = -\underbrace{\frac{1}{4\pi\varepsilon_0}}_{=k} \frac{e^2}{|\bar{r}_1 - \bar{r}_2|}$$

ELECTROSTATIC FORCE

(for electron-proton)

An isolated two body system is translationally invariant, the potential can only depend on

$$\bar{r} = \bar{r}_1 - \bar{r}_2$$

In addition, if we deal with a central conservative force, it is possible to show that the force can only depend on the magnitude of the distance between the two bodies, and not on the direction. This is precisely what happens in the cases of the gravitational and electrostatic forces. Therefore

$$U(\bar{r}_1, \bar{r}_2) = U(|\bar{r}|) = U(r)$$

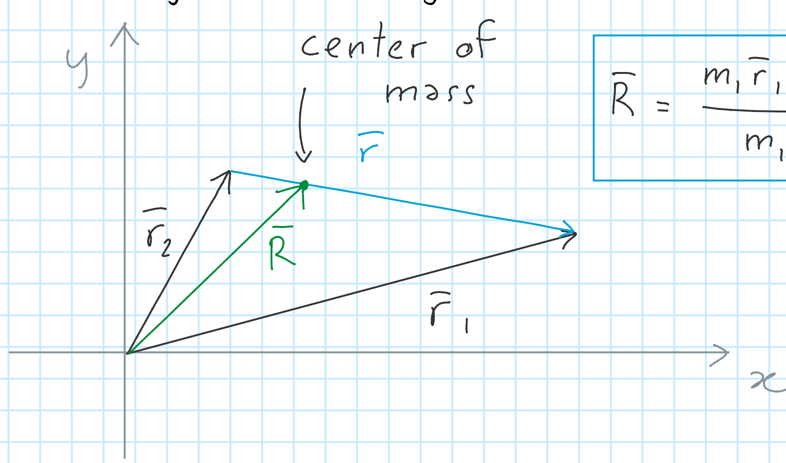
In practice, to study the two body problem in classical mechanics means to study the Lagrange's equations obtained from the Lagrangian

$$\boxed{\mathcal{L} = \frac{1}{2} m_1 \dot{\bar{r}}_1^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2^2 - U(r)}$$

TWO BODY LAGRANGIAN

## Reduced Mass and Center of Mass frame

It is convenient to locate the center of mass of the system and then rewrite the Lagrangian in terms of the position of the center of mass, R, and of the relative position of the two objects, indicated by r.




$$\boxed{\bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{m_1 + m_2}}$$

The center of mass will be located on the line joining the two particles. Indeed, if one chooses the frame of reference in such a way that the vector r is located on the x-y plane, the line joining the two particles has the equation

$$y = \frac{y_1 - y_2}{x_1 - x_2} x + \frac{y_2 x_1 - y_1 x_2}{x_1 - x_2}$$

It is straightforward to show that the coordinates of the center of mass satisfy the equation above

$$x_{cM} \equiv \frac{m_1 x_1 + m_2 x_2}{M} \quad , \quad y_{cM} = \frac{m_1 y_1 + m_2 y_2}{M}$$

rem

$$M \equiv m_1 + m_2$$

$$y_{cM} = \frac{y_1 - y_2}{x_1 - x_2} x_{cM} + \frac{y_2 x_1 - y_1 x_2}{x_1 - x_2}$$

It was shown before that the total momentum of the system is the momentum that a mass M moving with the velocity of the center of mass would have

$$\bar{P} = M \dot{\bar{R}}$$

Consequently, it will be useful to work in the center of mass rest frame, in which the origin of the frame is located at the center of mass of the system and the center of mass has zero velocity. Since the total momentum of an isolated system is conserved, in the center of mass rest frame the total momentum of the system will be zero.

For the time being let's aim to rewrite the Lagrangian in terms of r and R.

$$\bar{r} = \bar{r}_1 - \bar{r}_2 \qquad \bar{R} = \frac{m_1 \bar{r}_1 + m_2 \bar{r}_2}{M}$$

$$\bar{r}_2 = \bar{r}_1 - \bar{r}$$

$$M \bar{R} = m_1 \bar{r}_1 + m_2 \bar{r}_1 - m_2 \bar{r}$$

$$M \bar{R} = (m_1 + m_2) \bar{r}_1 - m_2 \bar{r} \qquad \boxed{\bar{r}_1 = \bar{R} + \frac{m_2}{M} \bar{r}}$$

$$\boxed{\bar{r}_2 = \bar{R} + \frac{m_2}{M} \bar{r} - \bar{r} = \bar{R} - \frac{m_1}{M} \bar{r}}$$

Therefore the kinetic energy can be rewritten as

$$T = \frac{1}{2} m_1 \dot{\bar{r}}_1^2 + \frac{1}{2} m_2 \dot{\bar{r}}_2^2 \longrightarrow$$

$$= \frac{1}{2}\left[ m_1 \left( \dot{\vec{R}} + \frac{m_2}{M} \dot{\vec{r}} \right)^2 + m_2 \left( \dot{\vec{R}} - \frac{m_1}{M} \dot{\vec{r}} \right)^2 \right]$$

$$= \frac{1}{2}\left[ M \dot{\vec{R}}^2 + \frac{\dot{\vec{r}}^2}{M^2} \left( m_1 m_2^2 + m_2 m_1^2 \right) \right]$$

$$= \frac{1}{2} M \dot{\vec{R}}^2 + \frac{1}{2} \underbrace{\frac{m_1 m_2}{M}}_{\equiv \mu} \dot{\vec{r}}^2$$

REDUCED MASS

The reduced mass is always smaller than both particle masses. If one particle has a much smaller mass than the other the reduced mass is very close to the smaller mass. The kinetic energy of the system is the same that a system of a particle of mass M located in R and a particle of mass $\mu$ located in r would have.

The Lagrangian of the two body problem can therefore be written as

$$\mathcal{L} = \underbrace{\frac{1}{2} M \dot{\vec{R}}^2}_{\mathcal{L}_{CM}} + \underbrace{\frac{1}{2} \mu \dot{\vec{r}}^2 - U(r)}_{\mathcal{L}_{rel}}$$

If one chooses to work in the center of mass rest frame the c.o.m. Lagrangian is simply zero.