Atwood machine with Lagrange multipliers

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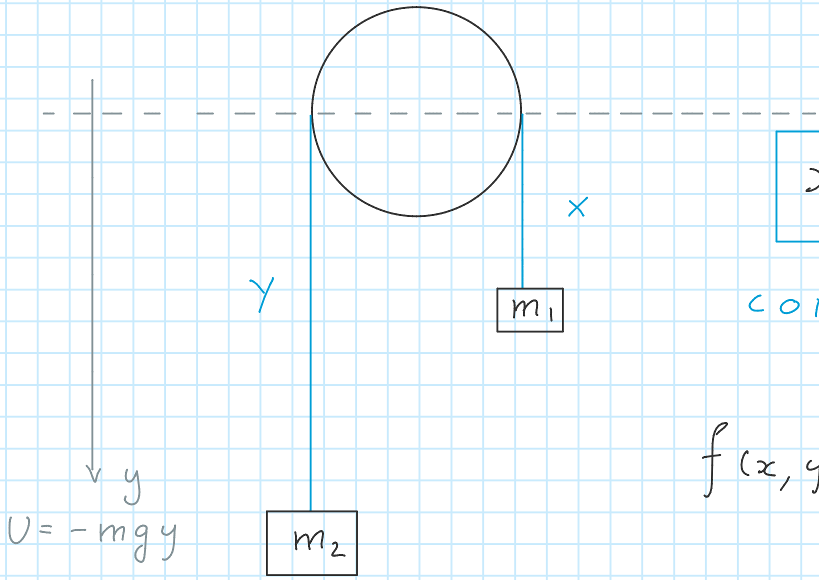
In order to show an explicit example of the use of Lagrange multipliers let's consider the Atwood machine. The vertical positions of the two masses are indicated by x and y respectively.



$$\boxed{x + y = \ell}$$

CONSTRAINT

$$f(x, y) \equiv x + y$$

$$\mathcal{L} = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{y}^2 + m_1 g x + m_2 g y$$

Consequently, the modified Lagrange equations are

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{x}} = m_1 g + \lambda - m_1 \ddot{x} = 0$$

$$\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} - \frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{y}} = m_2 g + \lambda - m_2 \ddot{y} = 0$$

In addition

$$x + y = \ell \longrightarrow \begin{cases} y = \ell - x \\ \ddot{y} = -\ddot{x} \end{cases}$$

Therefore the second Lagrange equation above becomes

$$m_2 g + \lambda + m_2 \ddot{x} = 0$$

$$\lambda = -m_2 g - m_2 \ddot{x}$$

substitute in the first eq

$$m_1 \ddot{x} = m_1 g - m_2 g - m_2 \ddot{x}$$

$$\boxed{\ddot{x} = g \frac{m_1 - m_2}{m_1 + m_2}}$$

EQUATION OF MOTION

When one wants to solve the problem from the point of view of Newton's second law, one needs to introduce the tension acting on the masses

$$m_1 \ddot{x} = m_1 g - T \qquad\qquad m_2 \ddot{y} = m_2 g - T$$

By comparing with the above, one sees that the tension is indeed equal to $-\lambda$. One can easily solve the equations

$$g \frac{m_1 - m_2}{m_1 + m_2} \equiv a \qquad\qquad x = \frac{1}{2} a t^2 + x_0$$

$$y = -\frac{1}{2} a t^2 + y_0$$

$$\lambda = -m_2 g - m_2 \ddot{x} = -m_2 (g + a) =$$

$$= -m_2 g \left(1 + \frac{m_1 - m_2}{m_1 + m_2} \right) = -\frac{2 m_1 m_2}{m_1 + m_2} g$$

$$\lambda = -m_1 g + m_1 \ddot{x} = m_1 (a - g) =$$

$$= m_1 g \left(\frac{m_1 - m_2}{m_1 + m_2} - 1 \right) = -2 \frac{m_1 m_2}{m_1 + m_2} g \quad \checkmark$$

The expressions for λ obtained from the first and second Lagrange equations agree with each other. The Lagrange multiplier is constant in this case.