Lagrange multipliers

Tuesday, September 17, 2019 8:36 AM

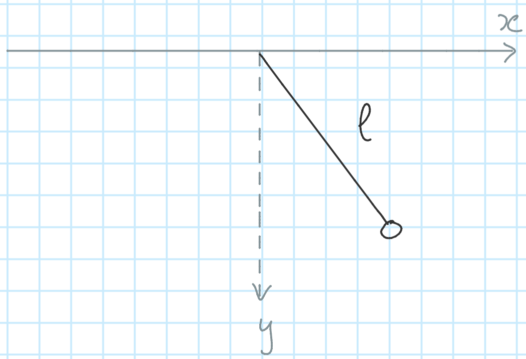
The method of Lagrange multipliers is a powerful method that allows one to fix implicit constraints and to figure out what are the constraint forces (in the case in which one needs to know them).

Taylor in his book discusses Lagrange multipliers by discussing the simple example of two-dimensional systems which only have one degree of freedom. Examples of these systems are the simple pendulum and the Atwood machine.

In general a two dimensional system with one degree of freedom will satisfy a constraint of the form

$$f(x,y) = \text{const.}$$

For example, in the simple pendulum



$$\sqrt{x^2+y^2} = \ell^2$$

By means of Lagrange's multipliers one can write the Lagrangian in terms of the cartesian coordinates and simultaneously impose the constraint. Even if one writes the Lagrangian in terms of cartesian coordinates rather than the generalized coordinates corresponding to the true degrees of freedom of the system, Hamilton's principle guarantees that the system will follow a trajectory that extremizes the action.

$$S = \int_{t_1}^{t_2} dt\, \mathcal{L}(x, \dot{x}, y, \dot{y})$$

THE ACTION IS STATIONARY

Therefore a small variation around the physical path results in a vanishing change for the action.

$$\left.\begin{aligned} x(t) &\longrightarrow x(t)+\delta x(t) \\ y(t) &\longrightarrow y(t)+\delta y(t) \end{aligned}\right\} \longrightarrow \quad \delta S = 0$$

However, δS can also be written as

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x}\,\delta x + \frac{\partial \mathcal{L}}{\partial \dot{x}}\,\delta \dot{x} + \frac{\partial \mathcal{L}}{\partial y}\,\delta y + \frac{\partial \mathcal{L}}{\partial \dot{y}}\,\delta \dot{y} \right) dt$$

Now observe that

$$\int_{t_1}^{t_2} \frac{\partial \mathcal{L}}{\partial \dot{x}}\,\delta \dot{x}\,dt = -\int_{t_1}^{t_2} \left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x\,dt + \underbrace{\cancel{\left(\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x \Big|_{t_1}^{t_2}}}_{\delta x(t_1) = \delta x(t_2) = 0}$$

Therefore

$$\delta S = \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x\,dt + \int_{t_1}^{t_2} \left(\frac{\partial \mathcal{L}}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y\,dt = 0$$

Now, if δx and δy were independent, one could prove that Lagrange's equations are valid for x and y separately. However, x and y are linked by the constraint and therefore their variations are not independent. The constraint requires that

$$\boxed{\delta f = \frac{\partial f}{\partial x}\,\delta x + \frac{\partial f}{\partial y}\,\delta y = 0}$$

Since the variation is zero we can add this to the integral that gives δS. In addition , one can even multiply f by a completely arbitrary function of t, which we indicate with $\lambda(t)$. The latter is the Lagrange multiplier

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial x} + \lambda(t)\frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} \right) \delta x\,dt + \int \left(\frac{\partial \mathcal{L}}{\partial y} + \lambda(t)\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} \right) \delta y\,dt = 0$$

Since λ is arbitrary, one can choose it in such a way that

$$\boxed{\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0}$$

MODIFIED LAGRANGE EQUATION FOR x

If this first relation is true, then, since $\delta S = 0$, one must also have

$$\boxed{\frac{\partial \mathcal{L}}{\partial y} + \lambda \frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{y}} = 0}$$

MODIFIED LAGRANGE EQUATION FOR y

One can then add the constraint equation to the two equations above and solve for x,y, and λ.

Notice that we could have obtained the modified equations starting from a modified Lagrangian. In fact, one can write the constraint as

$$g(x,y) = f(x,y) - \text{const} = 0$$

Since g vanishes, one can add it to the Lagrangian

$$\mathcal{L}' = \mathcal{L} + \lambda(t)\, g(x,y) = \mathcal{L} + \lambda\left(f(x,y) - \text{const}\right)$$

The modified Lagrangian above leads to the modified equations.

Lagrange multipliers and constraint forces

Consider the generic Lagrangian depending on x and y

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} m \dot{y}^2 - U(x,y)$$

$$\frac{\partial \mathcal{L}}{\partial x} + \lambda \frac{\partial f}{\partial x} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}} = 0$$

$$\boxed{-\frac{\partial U}{\partial x} + \lambda \frac{\partial f}{\partial x} - m\ddot{x} = 0}$$

However, we know that the only forces acting on the object besides the conservative forces described by U, are constraint forces. Therefore it must be

$$m\ddot{x} = \underbrace{-\frac{\partial U}{\partial x}}_{\text{conservative forces}} + \lambda \frac{\partial f}{\partial x}$$

FORCES OF CONSTRAINT